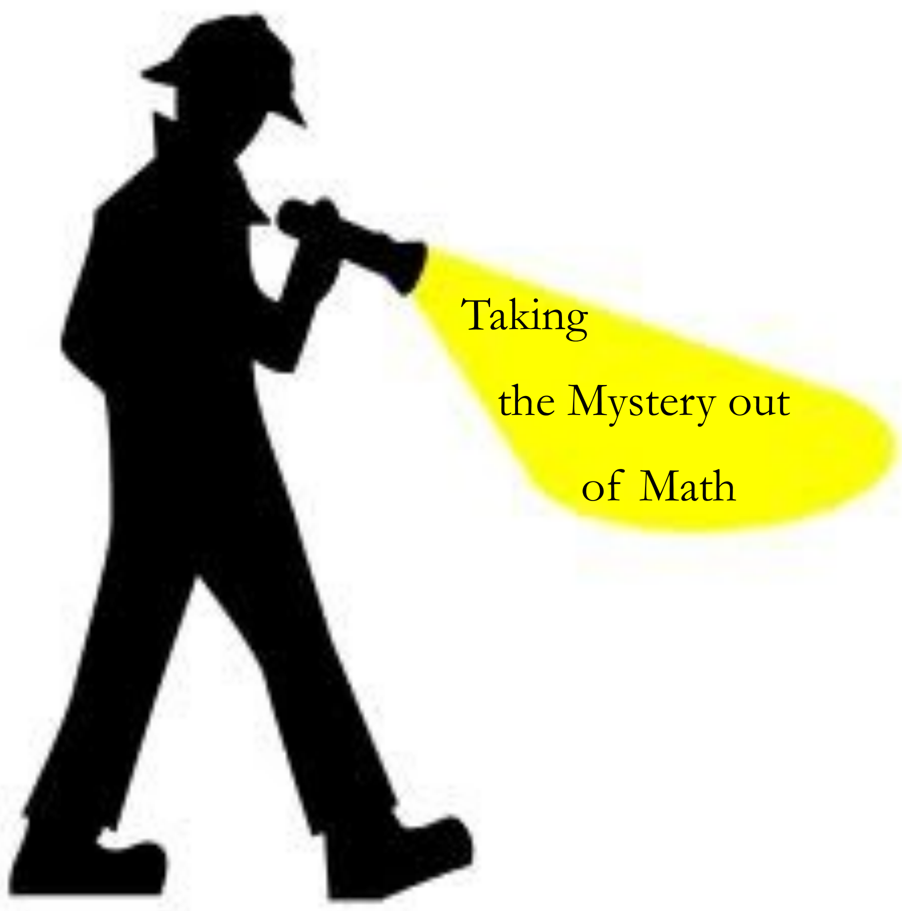
KATM Bulletin

Kansas Association of Teachers of Mathematics

Friday, October 16th

Maize High School

April 2015



KATM Fall Conference

Inside this issue:

A Message from our President

Spring is finally upon us. State Assessments are underway. Legislation that can impact what we do as teachers is pending. Yet the great teachers of Kansas teach on. We look out for our students and provide them with a quality education no matter what funding we have to work with. The purpose of KATM is to provide guidance to Kansas teachers so they can continue to provide quality math education. I am proud that our Bulletin editor and committee have produced the Bulletins with a theme each issue to help teachers better understand what the 8 Mathematical Practices look like in the classroom. This month's theme is Mathematical Practice #3 - construct reasonable arguments and critique the reasoning of others. This is a practice that reaches far beyond the math classroom and even one that our committee has put into practice while at the State House.

On Feb. 23, 2015, our Past-President Fred Hollingshead, and former board member David Barnes, testified in front of the House education committee in regards to House Bill 2292 which looks to repeal the Kansas College Career Ready Standards. I also submitted written testimony of behalf of KATM in opposition of this bill. A form of this bill has come up for debate the last several years. Each year, we hear the same concerns about the Common Core being written with no influence from Kansas Teachers. This is just not true. KATM has worked hard to critique the reasoning of others and construct viable arguments to not pass this bill. I thought it might be helpful for our members to be informed so I am publishing my letter of opposition to this bill.

Written Testimony before the
House Committee on Education on **HB 2292**
Development and establishment of K-12 curriculum standards.
By Stacey Bell, President, Kansas Association of Teachers of Mathematics
Instructional Coach
USD #450, Shawnee Heights Public Schools, Tecumseh, KS

19 February 2015

Chairman Highland, Members of the Committee;

Thank you for the opportunity to comment on **HB 2292**. I am providing written testimony today on behalf of KATM (Kansas Association of Teachers of Mathematics) in opposition of this bill on three grounds. First, the College and Career Ready Standards, also known as the Common Core, or the Kansas College and Career Ready Standards, were created with input from Kansas teachers. Second, these standards have been implanted for multiple years now and it has proven to be very successful in promoting critical thinkers, who have a deeper understanding of mathematics. Finally, Kansas students are developing valuable skills because these standards are not just teaching math, but also teaching them to think, reason, and persevere better than those of the last decade.

Created with a Kansas Influence

Just as your committee has heard for several years now, the College and Career Ready Standards were created with input from people in Kansas. We have had multiple board members from KATM serve on the Math Standards Review Committee. Some of these committee members have also testified to your committee in the last couple of years in regards to their involvement in providing input to insure that the standards they recommended to the Kansas State Board of Education would be the right choice for Kansas students. At no time was the review committee "required" to recommend these standards if they didn't feel it was in the best interest of Kansas children. These committee members were teachers with extensive expertise in mathematics education. It is vitally important for teachers to have a voice in the adoption of standards as they have been trained in math education. Experts in the field of education should be sought out and heard when making decisions about their area of expertise. We talk to doctors about our health, financial advisors about our money, and we should talk to teachers about what we should teach our children. Kansas teachers were consulted and heard not only in the development of the Common Core Standards, but also in the recommendation of these standards to the Kansas State Board of Education.

Implementation of Content Standards

As a reminder, many districts started the process of implementing the standards as early as 2010. In 2015, students all over Kansas have benefited from these standards and are proving to be much more successful in mathematics because they are required to think, reason, and persevere better than the students of the last decade. Since many students have been working with these standards for almost 5 years now, they have a much better conceptual understanding of how numbers work together. They are much stronger with their mental math skills and are much better problem solvers. These skills applied to the concepts of mathematics will produce productive citizens for Kansas. Teachers are noticing significant differences in their students' ability in computation skills as well as their applications of those computation skills in expressions, equations, the number system, ratios, proportionality, geometry, statistics, and probability

Legislation to reset curriculum and instruction would be difficult due to the fact that we aren't teaching different concepts, but we are just teaching them with a more purposeful and thorough approach and in some cases at a different grade level. Teachers have found more effective ways to teach their curriculum thanks to the new standards. If **HB 2292** would be passed, it would be asking teachers to go back 5 years and teach with much less effective teaching methods for the same content. We taught fractions in the old standards and we are still teaching fractions with the standards adopted in 2010. The timeline, methods, and strategies have changed to meet the most current research on how students learn mathematics.

Math content standards have been embedded into Kansas classrooms since the early 1990's when NCTM published their version of national standards for mathematics. Each state reviewed those standards and adapted them to create State Standards for mathematics. A similar process took place before the adoption of the KCCRS standards in 2010. Mathematicians and math educators developed standards and then passed them down to the states for review. Kansas educators have been implementing versions of these standards since the 1990's. The difference between those standards and the ones we are currently implementing is there is a focus on conceptual understanding first before we give kids rules to remember or numbers to memorize. There is a balance between understanding, procedure, and fluency throughout K-12.

Implementation on Math Practice Standards

Another difference in the standards is the inclusion of the 8 Mathematical Practice Standards. These standards help teach students to think, reason, and persevere when applying math concepts.

MP1 – Make sense of problems and persevere in solving them.
MP2 – Reason abstractly and quantitatively.
MP3 – Construct viable arguments and critique the reasoning of others.
MP4 – Model with mathematics.
MP5 – Use appropriate tools strategically.
MP6 – Attend to precision.
MP7 – Look for and make use of structure.
MPS — Look for and express regularity in repeated reasoning.

Through the last 5 years students have found success because math teachers have not only taught the content standards but also focused on the math practices as well. Our Kansas students are learning to think through problems and not give up at the first sight of adversity. They are taught to reason abstractly and quantitatively. They critique the reason of others and practice error analysis. They make models to better understand the math problems. They use appropriate tools to solve the problems. They attend to precision and determine to what level they need to be accurate. Do they estimate or is an exact answer needed? They also are asked to use precise math language. They look for patterns and apply those patterns to solve problems. Finally they look for patterns over time. All of these 8 mathematical practices combined with the content standards set Kansas students up for success far past their K-12, college, and or career. These are standards we want for Kansas students.

Conclusion

KATM opposes **HB 2292** because 1.) Kansas teachers helped review the standards, provided feedback to the authors, and recommended these standards for adoption to our Kansas State Board of Education. 2.) Teachers all over the nation are able to network, share resources, and share approaches to teach students with the most effective math strategies because we all have a common goal and have been so for 5 years. Due to the success we have seen with our students, if this bill were to pass, I am confident that the new standards required to be developed in 2017 would be very similar to what we are currently using today. Kansas teachers are excited about teaching math and are much more knowledgeable about how to teach math at a level needed for students to understand why the math works instead of just teaching the rules and asking them to memorize numbers thanks to the adoption of these standards in 2010. 3.) These standards are producing students that are ready to enter college, technical colleges, or careers more so than ever before with reasoning skills and the ability to persevere. More importantly, students are feeling successful. They finally understand math and want to learn more. Parents are amazed at what their kids can do and understand compared to their own experiences, as well as the understanding of their older siblings. My own fourth grader can divide four and five digit numbers by two numbers all in his head due to his increased number sense thanks to the conceptual understanding and math models used to teach that concept. I taught many 7th graders for years that couldn't solve a problem like this in their heads nor would they even attempt the problem. For these reasons, KATM asks that you vote in opposition of this bill and not let it out of committee.

Respectfully submitted,

Stacey Bell

Stacey Bell, KATM President

KATM Testifies to House Ed Committee

Fred Hollingshead, Past-President, Community Relations, testified in front of the House Education Committee in opposition of HB 2292, this year's version of the bill seeking to abolish our state math standards adopted in 2010. He and Board President Stacey Bell also provided written testimonies. Some version of the bill has appeared in committee each of the last three years. This is also the third year KATM has provided both written and in-person testimony against the legislature's attempts to circumvent established policies for curriculum standards reviews.

The House Committee on Federal and State Affairs introduced the bill without any specific legislators attaching their name to it. The Committee is one of the few which can have its bills acted on after "turn around," the date when action in most other committees (including the Education Committee) is complete. This fact is important, as it would allow members to still review and take action later, perhaps when fewer people would be paying attention. This is yet another example of the House playing political games with public education and Kansas children.

This year, HB 2292 attempted to go beyond merely banning our math standards. The bill sought to remove any standards adopted since October 12, 2010, and reinstate those standards adopted in 2003. This not only would have voided the work of math educators over the last five years, but also in other contents such as language arts, science, social studies, and career/technical education, among others. Additionally, the bill would have banned any assessments created to assess those sets of standards. What the legislators failed to consider is how wide-reaching such a bill be. The ACT, MAP, CTE, AP, and other high stakes assessments have all been realigned to the new standards we adopted, and the law would have forbidden our students from taking any of them.

HB 2292 would have also disallowed any person or agency representing Kansas to work with others to develop standards (apparently, the House feels collaboration will lead to poorer standards). The bill would have also prevented districts, schools and teachers from spending any public *or* private money on any materials aligned with these sets of standards. Again, such action would mean even parents who homeschool their children would not be allowed to buy most of the resources currently available. Finally, the bill addressed numerous other issues not specifically related to our standards (like data, teacher evaluations, the formation of SITE councils.

We continue to oppose any attempts of action brought to the capitol which would undo the recent years of hard work and progress we have made. Over the past three years, your Board has advocated on your behalf by appearing in front of the House Education Committee as well as through submitted written testimonies. This year, following a hearing on February 23rd, Committee Chair Ron Highland (R-Wamego) made it clear there would not be enough support in the full House to pass the bill and planned to let it die in committee. Nearly a month later, he did allow it to come up for action, and after multiple attempts to amend the bill, it failed after an apparent 9-6 vote (note this is an unofficial tally – no roll call vote was taken and there are 19 members on the committee).

Hello Kansas Math Teachers!

I hope this issue of the KATM finds you well, and looking forward to a restful summer! When I first heard about the standards for Mathematical Practice, I was a little bit overwhelmed. Some of them, however, seemed easier to wrap my mind around than others. #3 was a practice that I could more easily imagine what it would look like in my classroom. Hopefully, this issue will give you lots of good ideas of how to use this practice in your classroom.

This month you will notice several strategies adapted from ideas in the book *Mathematics Formative Assessment: 75 Practical Strategies for Linking Assessment, Instruction and Learning.* This great resource by Page Keeley and Cheryl Rose Tobey has tons of great ideas. The strategies can be adapted to a wide variety of grade levels, content standards and math practice standards. Hopefully, you will find the adaptations in this Bulletin to be useful, and if you're looking for something to add to your summer reading list, this may be the book!

Happy Reading!

Jenny Wilcox

KATM Bulletin Editor

Focus Issue: SMP #3 Construct viable arguments and critique the reasoning of others

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. (from corestandards.org)

What does this look like K-5?

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. For example, a student might argue that two different shapes have equal area because it has already been demonstrated that both shapes are half of the same rectangle.

They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. For example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor.

They justify their conclusions, communicate them to others, and respond to the arguments of others. Students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8, Look for and express regularity in repeated reasoning).

They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. For example, in order to demonstrate what happens to the sum when the same amount is added to one addend and subtracted from another, students in the early grades might represent a story about children moving between two classrooms: the number of children in each classroom is an addend; the total number of children in the two classrooms is the sum. When some students move from one classroom to the other, the number of students in each classroom changes by that amount—one addend decreases by some amount and the other addend increases by that same amount—but the total number of students does not change. An older elementary student might use an area representation to show why the distributive property holds. (continued on next page)

Focus Issue: SMP #3 Construct viable arguments and critique the reasoning of others

Such arguments can make sense and be correct, even though they are no generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. For example, young students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands. In upper elementary grades, students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals. For example, students might make an argument based on an area representation of multiplication to show that the distributive property applies to problems involving fractions.

Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Illustrative Mathematics. (2014, February 12). *Standards for Mathematical Practice: Commentary and Elaborations for K–5.* Tucson, AZ.

What does this look like 6-8?

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures For example, students might conjecture that the diagonals of a parallelogram bisect each other, after having experimented with a representative selection of possible parallelograms.

They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. An important use of counterexamples in Grades 6–8 is the use of numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that $5 - 2x$ is equivalent to $3x$.

They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. For example they might argue that the great variability of heights in their class is explained by growth spurts, and that the small variability of ages is explained by school admission policies.

'Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and – if there is a flaw in an argument – explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Proficient middle school students progress from arguing exclusively through concrete referents such as physical objects and pictorial referents, to also including symbolic representations such as expressions and equations.

Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Illustrative Mathematics. (2014, May 6). *Standards for Mathematical Practice: Commentary and Elaborations for 6–8.* Tucson, AZ.

How Can I Develop Practice 3 in my Classroom?

What type of task will help?

Is structured to bring out multiple representations, approaches, or error analysis.

Embeds discussion and communication of reasoning and justification with others.

Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.

Expects students to give feedback and ask questions of others' solutions.

What should the teacher do?

Create a safe and collaborative environment.

Provides **ALL** students opportunities to understand and use stated assumptions, definitions, and previously established results in constructing arguments.

Provides ample time for students to make conjectures and build a logical progression of statements to explore the truth of their conjectures.

Provides opportunities for students to construct arguments and critique arguments of peers.

Facilitates and guides students in recognizing and using counterexamples.

Encourages and facilitates students justifying their conclusions, communicating, and responding to the arguments of others.

Asks useful questions to clarify and/or improve students' arguments.

Provide time and value discourse

What should students do?

Question others.

Support beliefs and challenges with mathematical evidence.

Form logical arguments with conjectures and counterexamples.

Recognize and use counterexamples.

Justify and defend ALL conclusions and communicates them to others.

Created by Melisa J. Hancock (melisa@ksu.edu) CCSS-M Mathematical Practices, Park City Math Institute-Rigor Rubric, 2013.

Classroom Discussion Sentence Starters

What to say to explain what you did:

I learned that I...
I was surprised that I...
I noticed that I...
I discovered that I...
I started by…
I think that… and here is my reason…

What to say when you agree with the ideas of others:

My idea is related to _____'s idea
I really liked ______'s idea about
I resonate with what ______ said
You made a great point about
I hadn't thought about that
My idea builds on _____'s idea
I'd like to piggy back on _______

What to say when you disagree

Then again we shouldn't forget
I see it differently. Based on
That's a valid point but I feel
I understand the idea of… but I feel that…
On the other hand
I do agree with _____ but I disagree that ______
True but what about

What to say when you want clarification

Can you elaborate on that?
In other words are you saying…
I'm not quite clear, can you explain the part about… again
I have a question about
Do you mean that
Can you clarify the point about _____ for me?

Other discussion starters

What mathematical evidence would support your solution?

How can we be sure that...?

How could you prove that...?

Will it still work if...?

What were you considering when...?

How did you decide to try that strategy?

How did you test whether your approach worked?

How did you decide what the problem was asking you to find? (What was unknown?)

Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not?

How do you create a classroom culture for critiquing the reasoning of others???

A classroom activity to teach critiquing, adapted from http://themathymurk.blogspot.com/

This was on the board to give students specific ideas of what was expected:

First Draft ≠ Final Draft

Feedback Expectations

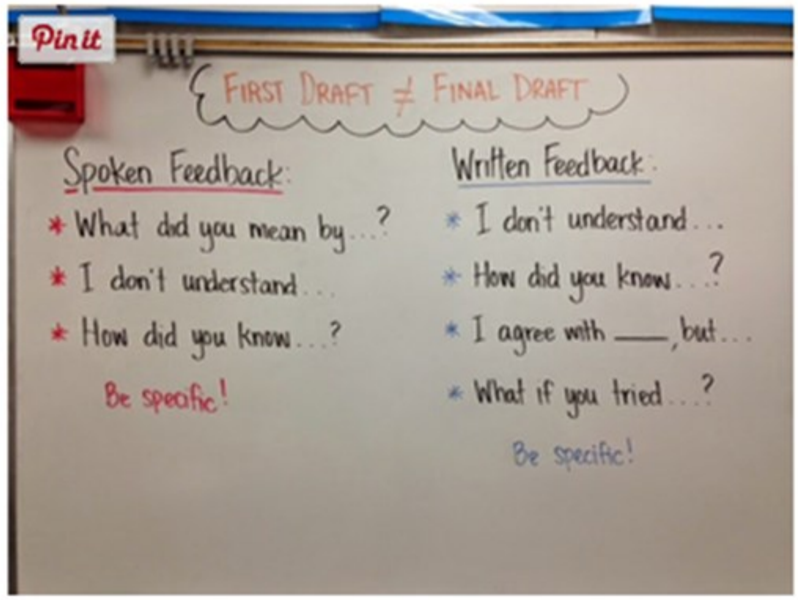
*Be specific

*Avoid opinions

*Think about what feedback you would find helpful

*Giving feedback ≠ being mean

*It takes practice! We will work through it together!



For the lesson, students were given a task that required written justification of reasoning. Students then turned to a partner, and read their argument out loud. The partner had to provide verbal feedback about the argument. Students were then given a chance to revise the argument based on the verbal feedback. Then students switch roles. Finally, the students choose one of their arguments that they will share with another pair.

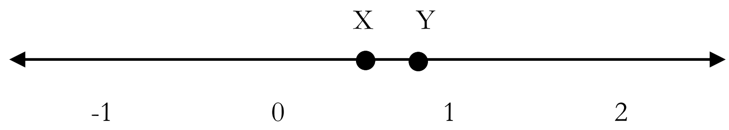
At this point, take a few minutes to look at a same piece of student week and brainstorm what kind of feedback would be useful to improve the argument. Now the pair exchanges arguments with another pair. This time, the pairs will give written feedback on the arguments. You could go through this written feedback process one more time, if desired.

Now the papers are returned to the original owners, who get a chance to revise the argument.

When the arguments have been revised, you can also share different approaches with the class.

On the next page, you will find a sample worksheet with a task that could be used for this activity. The worksheet is easily adapted to different grade levels and topics simply by changing the prompt at the top.

Task: On the number line below, mark and label the approximate position of Pif P = X + Y. Justify your answer in the box below.



First Draft of Justification:

Justification After Verbal Feedback:

Feedback to improve clarity, viability and logic:

Feedback to improve clarity, viability and logic:

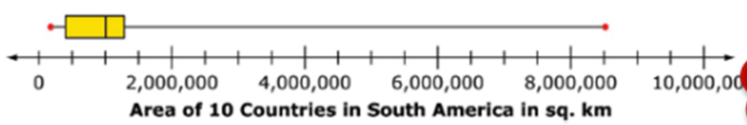
Revised Version of Justification:

These SBAC sample items show different types of questions that focus on math practice #3.

The areas, in square kilometers, of 10 countries in South America are shown in the table.

Country	Area, in Square Kilometers
Uruguay	176,215
Ecuador	256,369
Paraguay	406,752
Chile	756,102
Venezuela	912,050
Bolivia	1,098,581
Colombia	1,141,748
Peru	1,285,216
Argentina	2,780,400
Brazil	8,514,877

The data is also summarized in the box plot.



Which measure of center, the mean or the median, is best to use when describing this data? Thoroughly explain your reasoning for choosing one measure over the other measure.

Grade 6
Claim 3
DOK 3

Justify your answer

Two of these statements are true in **all** cases:

- Statement 1: The greatest common factor of any two distinct prime numbers is 1.
- Statement 2: The greatest common factor of any two distinct composite numbers is 1.
- Statement 3: The product of any two integers is a rational number.
- Statement 4: The quotient of any two integers is a rational number.

Part A: Which two statements are true in all cases?

Part B: For both statements that you did not choose in *Part A*, provide one clear reason and/or example for each statement that proves the statement can be false.

Statement ☐ Reason/example

Statement ☐ Reason/example

Use a counter-example to justify your reasoning

Grade 7
Claim 3
DOK 3

Go to http://sbac.portal.airast.org/practice-test/, to fine practice tests available for grades 3-HS.

Ashley and Brandon have different methods for finding square roots.

Ashley's Method

To find the square root of x, find a number so that the product of the number and itself is x. For example, $2 \cdot 2 = 4$, so the square root of 4 is 2.

Brandon's Method

To find the square root of x, multiply x by $\frac{1}{2}$. For example, $4 \cdot \frac{1}{2} = 2$, so the square root of 4 is 2.

Which student's method is **not** correct?

- ☐ Ashley's method
- ☐ Brandon's method

Explain why the method you selected is **not** correct.

Explain flawed logic of others

Grade 8
Claim 3
DOK 3

Mr. Perry's students used pairs of points to find the slopes of lines. Mr. Perry asked Avery how she used the pairs of points listed in this table to find the slope of a line.

x	*y*
8	18
20	45

Avery said, "The easiest way to find the slope is to divide y by x. The slope of this line is $\frac{18}{8}$, or $\frac{9}{4}$."

Part A

Show another way to find the slope of the line that passes through the points listed in the table. Your way must be different from Avery's way.

Part B

Write an example that shows that Avery's "divide y by x" method will not work to find the slope of **any** line.

Explain flawed logic with a counter example

Grade 8
Claim 3
DOK 2

PARCC also has sample tests available for grades 3-HS, at http://parcc.pearson.com/practice-tests/math/

WOW! Mathematics Convention: A Community Connection

Rebecca R. Cavazos

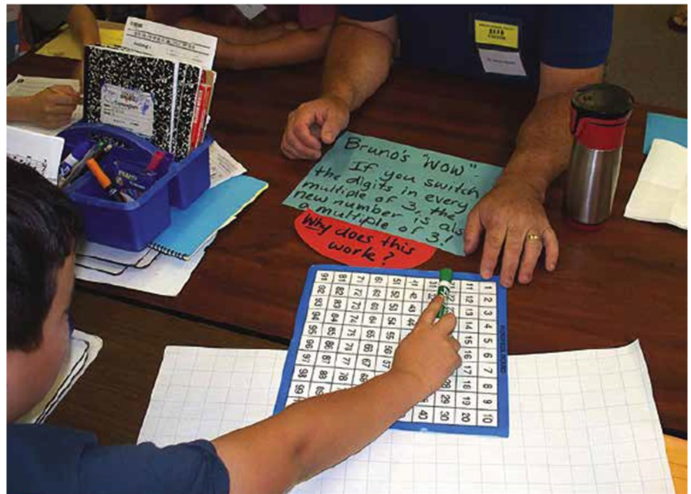
"By asking questions, noticing patterns, and just plain thinking about things that make people say 'Wow!' humanity adds every day to the list of amazing things we've figured out. There are lots of mysteries out there that no one understands, and still more that no one even knows about yet. You never know what new knowledge will come in handy in the future, or who will use that knowledge differently than ever before to discover something new. So keep up the great work, keep your eyes open for amazing things in school and all around you, . . ." (from an e-mail to students by Andrew Marble of the National Solar Observatory)

Dr. Marble (the father of a student in my class), was one of our community participants at our First Annual WOW! Mathematics Convention for my fourth-grade class at Borton Magnet School in Tucson, Arizona. Also joining us were a math professor, a biologist, a literacy professor, a chemist, a statistician, and an engineering student as well as our school principal and computer lab technician, both of whom are math enthusiasts. This article details how certain mathematical "discoveries" that my fourth graders made were recorded throughout the year and then investigated intricately within a "convention" involving STEM experts from the community. My intent is to share with you one way that I successfully integrated the community into my classroom, which ultimately benefited community members and students alike.

Cavazos invited eleven community STEM experts to a classroom "math convention." After introducing them to her fourth graders, Cavazos asked the visitors to talk about how they use mathematics in their jobs.

We called on STEM experts in our community to help my fourth-grade students figure out the "why" behind certain patterns in numbers that they noticed throughout the year. As we progressed through our Investigations in Number, Data, and Space curriculum (TERC 2008), I would take note of some of the discoveries that students made. The students were accustomed to looking critically at mathematical circumstances as well as taking risks in communicating their ideas. They know that the mathematical standards of Reasoning and Proof, Problem Solving, and Communication (as defined by NCTM 2000) are an integral part of learning in our classroom. The mathematical discourse orchestrated in the classroom on a daily basis is evidence of my belief in the importance of the Standards for Mathematical Practice (SMP) in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010). I regularly ask my students to explain their thinking, to prove their answers, and to justify their reasoning. I also ask students to listen to others' reasoning, to add to it, to agree or disagree with it, and then to justify their position.

Yearlong discoveries



Bruno's WOW! set a precedent early in the academic year for the teacher to record and post fourth graders' observations of patterns that the class could not explain.

The first discovery that surfaced was Bruno's. When we initially explored factors and multiples, he studied my classroom hundred chart when the multiples of three were covered by a colored transparency and noticed that when he reversed the digits in the multiples of three, the new number was also a multiple of three. I had honestly never thought about this, so my reaction was, "Wow, you're right!" We took some time to see if it really worked for all multiples of three and to try to understand why. It worked for multiples through ninety-nine, but we never got to the answer for why. I decided to record our investigation on construction paper, call it "Bruno's WOW!" and leave it posted in my classroom for further investigation. This prompted my students to look for WOWs throughout the rest of the year. A WOW! became an observation by a student of a pattern that works in multiple cases but could not be readily explained in the timeframe of the lesson or within students' (and sometimes the teacher's!) mathematical abilities. Naming WOWs with students' names was great incentive for them to look for these functional relationships in math. We subsequently accumulated a total of four WOWs. We posted these throughout the year, and students who finished their math work early were encouraged to explore the WOWs to determine why they work.

A math convention

Toward the end of the school year, after completing the state testing, we decided to hold a WOW! Mathematics Convention to see if mathematicians in the community could come in and help us solve the "whys" of our WOWs. I put out a memo to parents. I had several University of Arizona parents of students in my current class. I have also hosted many student teachers and preservice teachers in my classroom during the past few years, so I connected with the education professors at the university. I invited a student majoring in engineering (my son), our principal, and our computer lab technician. Eleven adult math supporters worked with twenty-three student math fanatics who were ready to problem solve. I also had secured a cheat sheet from a retired physicist living in California. He had wanted to participate via Skype, but our technology deficiencies prohibited his live participation. However, he emailed written explanations, mostly algebraic, to explain why the WOWs work.

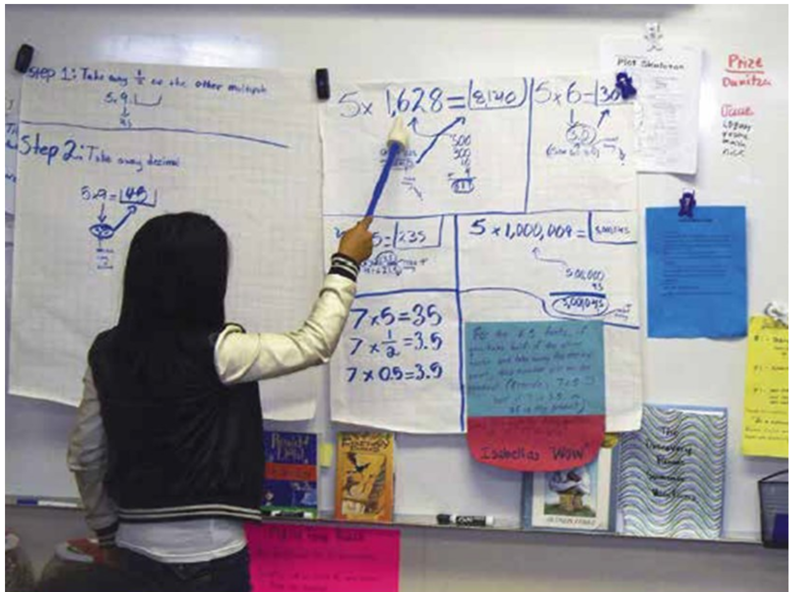
Group discourse

After introducing our STEM guests and having them tell a little about their jobs and how they use math at work, I divided the class into four groups and assigned a different WOW! to each. The adults went to the group that had the WOW! with which they were most comfortable. I made available hundred charts, three-hundred charts, number lines, white boards, and chart paper. I had planned for the convention to last about an hour, but the mathematics discourse was still going strong after an hour. Most students were completely engaged the entire time. I had wanted time for the groups to share their discoveries, so I decided to encourage each group to wrap up their discussions and plan how they would present their findings to the rest of the group. Students then shared what they had discovered and were much invested in their work as they attempted to explain the reasoning behind their WOW.

To show the type of rich dialogue and deep thinking that took place, I will attempt to provide a synopsis of what transpired in a couple of the groups. Isabella's WOW! was stated as such:

> For the times-five facts, if you take half of the other factor and take away the decimal point, this number will be the product. (E.g., $7 \times 5 = n$Half of 7 is 3.5, so 35 is the product.)

Students in this group were primarily coached by the engineering student and the biologist. They started out by testing different smaller numbers to prove that this theory would hold true for all numbers. The students were excited to show the adults that it was a sound theory. The adults encouraged them to list multiples of five in an organized way. The adults also prompted students to think about the multiples of ten and find the relationship between the two lists. They asked the students what they knew about five and ten. This discussion led to the students' understanding of why you can take half of the number being multiplied by five, eliminate the decimal point, and have the product. Students were eager to try much larger numbers. They called me over at one point to exclaim that their theory would work even for infinity! But they went on to prove for me that it worked for 5 × 1628 by halving the factor. The answer, they stated, was 814.0 because "we're using decimals."



Mariel explained why, when multiplying by fives, you can take half of the other factor, delete the decimal point, and find the product. Almost every student in this group was able to explain as well.

Mariel stated confidently, "You just take away the decimal, and you have your answer!"

The group also proved that it worked for 1,000,009, which was really stretching their field of number sense. At this point, the group had a large piece of chart paper with lots of figuring on it. The adults helped them create an organized chart of the steps they had taken to explore their WOW! This exercise provided reinforcement for those in the group who might have still been unsure about their findings. When sharing with the class, Isabella began with an explanation of what she was thinking when she came up with her WOW! and the fact that it works for all multiples of five. With a visual representation, Mariel continued to confidently show how it works:

$7 \times 5 = 35$

$7 \times 1/2 = 3.5$

$7 \times 0.5 = 3.5$

The group was asked, "Why does it work?" and Josue responded that five is half of ten, and ten is the basis of our number system. Although he did not exactly articulate the relationship between multiplying by ten and then dividing by two in the presentation, it was clear that the group had at least a partial understanding of the base-ten system. They demonstrated on their chart that their theory worked even for larger numbers. Almost every student who participated in this group felt confident enough in their mathematical abilities to explain their findings without the help of the adult mathematicians. Plus, they created a practical application of decimal and fraction use and the need to multiply larger numbers. The fourth-grade Investigations curriculum does not go much beyond multiplying double-digit numbers.

In contrast, Bruno's group was unable to explain their WOW! in the time provided. They were encouraged by the STEM expert working with them to look at the difference between the 2 two-digit numbers that were multiples of three. For example, eighteen and eighty-one are both multiples of three. The difference between these two numbers is sixty-three, which is also a multiple of three. This, they found, works for all two-digit multiples of three. When they began to explore three-digit multiples of three, something new happened. The *sum* of the digits in such a number was always a multiple of three. They also surmised that these rules apply for negative numbers and thus created a negative hundred chart and taped it to the positive hundred chart to prove their theory. They were able to continue the diagonal pattern of multiples of three that they had observed on the hundred chart onto the negative hundred chart. They also discovered that the negative hundred chart somewhat mirrors the positive chart, as happens on a number line. In class, we had explored negative numbers only using a number line. Needless to say, this group explored a lot of math. Bruno's group members were unable to put together much of a visual to explain the WOW! but they were able to demonstrate the negative and positive hundred charts and what happens with the digits in a multiple of three. They delved into some practical algebraic concepts (using letters as placeholders of numbers) when demonstrating what happens with three-digit multiples of three. Because of the number of concepts with which this group was able to go into depth, I believe the "convention" format was just as valuable as it was to the other groups, who *could* explain their WOWs.

Bruno's group created a negative hundred chart to prove that the multiples-of-three WOW! worked for both negative and positive numbers, but the students ran out of class time before they could fully explain all the in-depth mathematics they had explored.

After the groups had shared their discoveries, I also shared what I had observed that day, in addition to dusting off the old cobwebs of some algebra concepts. In my opinion, solving the math problems was not the real thrill of the day, as some of the concepts explored were still a little out of reach for many of the students. What was exciting was all the math that was grappled with in that hour-and-a-half. So many "mysteries" in the world of numbers can be pondered, explored, and possibly solved with a little perseverance. After two hours of intense brain exercise, we served lemonade and cookies as we thanked our community members for participating.

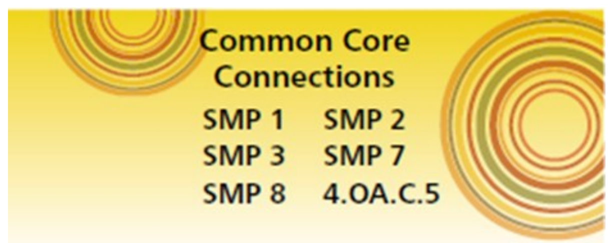
Meeting multiple goals

In addition to accomplishing the goal of having my students work to solve our WOWs with community members who use math in their careers, I believe I accomplished two more goals. First of all, it was evident that my students felt empowered by working with the adults and then sharing their findings. Throughout the process, students were treated as equals and felt like they had an integral part in the potential solutions. When they shared their findings, they just glowed with ownership and displayed confidence in their group's discoveries. Only a few students were not fully invested in their mathematical results, yet they all saw themselves as mathematicians.

The second outcome of this convention had more to do with the community's perspective of our modern -day classroom. During the convention, my students effectively modeled the academic discourse that takes place all year long in my classroom. This discourse, which is vital to the integration of CCSSM, is often unfamiliar to the general public. Many adults expect a mathematics classroom to resemble those with which they were schooled. I believe it is important for our community to understand how the rigor of CCSSM and in-depth investigations must yield a different type of learning community if we are to have students who are college and career ready. For our adult participants, students' use of critical thinking, collaboration, and articulation in the final presentations was an excellent prototype of what our twenty-first–century classrooms need to look like under CCSSM.

An example of such discourse was apparent in Taylor's group's discussion about her WOW. Taylor explained how the sum of the two digits for the multiples of nine always adds up to nine. Then another child in the group realized that beyond 9 × 10 yields a product that does not fi t the rule. At that point, Maya hypothesized that maybe the sum of the digits is a multiple of nine. For example, 9 × 11 = 99, and 9 + 9 = 18, which is a multiple of nine. The adult, a university math professor, then suggested that students start a written list of combinations beyond 9 × 10. The group members continued to explore the list to make conjectures of their own, and subsequently presented their findings to the audience at the convention.

Community connections



Having community members infiltrate my math classroom was beneficial not only for the students' growth but also for the adults' understanding of classroom practices, which are transforming to meet the needs of the twenty-first–century learner. Holding this type of "convention" was a risk to take on my part, because I really did not know how the adults would interact with the students. I was unsure whether they would tell students why the WOW! worked or if they would help lead the students to discovering it for themselves. The STEM community proved to be a valuable resource in validating my students' mathematical thinking and empowering them to go beyond what they deemed themselves capable of. My hope is that the WOW! Convention was a seed that will lead to further community connections across grade levels in the coming school year. As Crystal Kalinec-Craig of the University of Arizona stated,

> The WOW! Convention was evidence that classrooms can be safe spaces for children, families, and community members to take risks in terms of thinking about conceptually challenging ideas in mathematics.

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The standard in elementary school

As every teacher knows, children love to talk. But *explanation*—clear articulation of a sequence of steps or even the chronology of events in a story—is very difficult for children, often even into middle school. To "construct a viable argument," let alone understand another's argument well enough to formulate and articulate a logical and constructive "critique," depends heavily on a shared context, especially in the early grades. Given an interesting task, they can *show* their method and "narrate" their demonstration. Rarely does it make sense to have them try to describe, from their desks, an articulate train of thought, and even more rarely can one expect the other students in class to "follow" that lecture any better than—or even as well as—they'd follow the train of thought of a teacher who is just talking without illustrating. The standard recognizes this fact when it says "students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions." The key is not the concreteness, but the ability to situate their words in context—to show as well as tell.

To develop the reasoning that this standard asks children to communicate, the mathematical tasks we give need depth. Problem that can be solved with only one fairly routine step give students no chance to assemble a mental sequence or argument, even non-verbally. The inclination to "justify their conclusions" also depends on the nature of the task: certain tasks naturally pull children to explain; ones that are too simple or routine feel unexplainable. Depending on the context, "I added" can seem to a child hardly worth saying. And finally, *skill* at "communicating [a justification] to others" comes from having plentiful opportunities to do so. The way children learn language, including mathematical and academic language, is by *producing* it as well as by hearing it used. When students are given a suitably challenging task and allowed to work on it together, their natural drive to communicate helps develop the academic language they will need in order to "construct viable arguments and critique the reasoning of others."

One kind of task that naturally "pulls" children to explain is a "How many ways can you…" task.

> *How many ways can you make 28¢? (Variant: How many ways can you make 28¢ without using dimes or quarters?)*
>
> *How many different 5"-tall towers of 1" cubes can be made, using exactly one white cube and four blue cubes? (Variant: How many different 5"-tall towers of 1" cubes can be made, using exactly two white cubes and three blue cubes?)*

The first time young students face problems like these, they tend to be unsystematic. But after they have worked problems like this two or three times, they tend to develop methods (not necessarily efficient or correct, though often so). Then, faced with the question "How can you be sure there are no more?" most children, even as young as second grade, are drawn to explain and do so readily.

Similarly, in the playful context of an imaginary island with two families—one that always tells the truth, and one whose statements are always false—students can hardly stop themselves from explaining how they get answers to questions like this:

> *You meet Adam and Beth. Adam says "We're both from the family of liars." Which family is Adam from? What about Beth?*

Children (and adults) typically find it far easier to solve the puzzle than to say how they solved it, but it's also typical for them, given the slightest "how'd you get that?," to feel compelled to explain!

While young students can sometimes detect illogical arguments, it is not generally sensible to ask young students to critique the reasoning of others, as it is often too hard for them to distinguish flaws in the logic of another student's argument from artifacts created by the difficulty all young students have in articulating their thinking without ambiguity.

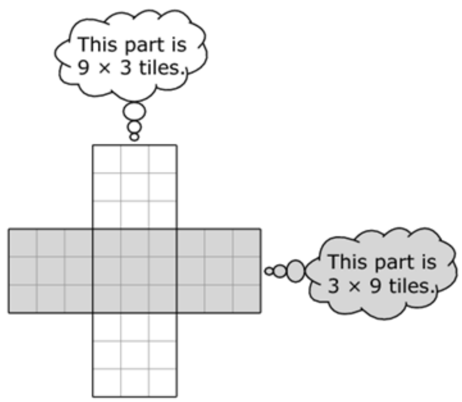
Elementary Practice #3 Activity

"Commit and Toss" is a strategy described in the book *Mathematics Formative Assessment: 75 Practical Strategies for Linking Assessment, Instruction and Learning* by Page Keely and Cheryl Rose Tobye. This strategy begins with students answering a multiple choice question about a content standard. Students not only choose an answer, but explain their reasoning for choosing that answer. Students do NOT write a name on the paper. Next, students crumple up their paper, and the fun begins! Students toss the paper balls around the room until the teacher tells them to pick one up and keep it.

At this point, there are several options in this activity. The teacher may choose to have different students share what is written on the paper. Since the response is anonymous, it is low-stress for students. The teacher might choose to have students go to four corners of the room, so that students could see how the answers were distributed. Students in each corner could then compare responses and critique the reasoning given.

This activity can be an effective way to spark discussion, or find out about student prior knowledge leading into a lesson. Below is an elementary example of "Commit and Toss" based on a Smarter Balanced practice item for 3rd grade.

Tasha is doing an art project with square tiles. She needs to figure out how many tiles she will need. This picture shows her design. Tasha thinks:



Tasha says, "I need $(9 \times 3) + (3 \times 9) = 27 + 27 = 54$ tiles to make the design."

Which statement explains why Tasha is **not** correct?

Ⓐ $27 + 27$ does not equal 54.

Ⓑ (3×9) does not equal (9×3).

Ⓒ Tasha multiplied 9×3 incorrectly.

Ⓓ Tasha included the 9 squares in the middle twice.

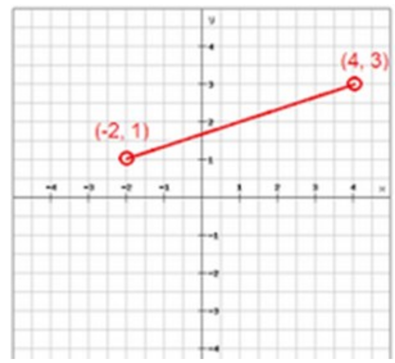
Explain your reasoning:

High School Practice #3 Activity

"Strategy Probe" is a strategy described in the book *Mathematics Formative Assessment: 75 Practical Strategies for Linking Assessment, Instruction and Learning* that has students complete an activity, and then compare several different methods for correctly solving the same task. Students must make sense of other students' solution processes, as well as compare them to their own.

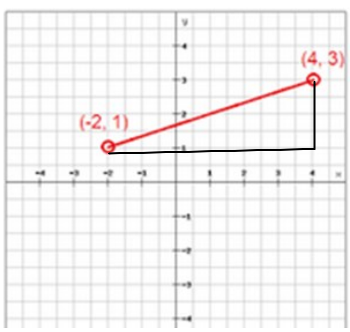
What's Your Distance Strategy?

Julie and Nate each found the distance between the points. Circle the method that is most similar to the way you would solve the problem.



<u>Julie's Method</u>

I thought of the distance as the hypotenuse of a right triangle and then used the Pythagorean Theorem.



Leg A = $|-2-4| = 6$

Leg B = $|1-3| = 2$

$A^2 + B^2 = C^2$

$6^2 + 2^2 = C^2$

$40 = C^2$

= C

$\sqrt{40}$

<u>Nate's method</u>

I used the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\sqrt{(-2-4)^2 + (1-3)^2}$$

$$\sqrt{(-6)^2 + (-2)^2}$$

$$\sqrt{36+4}$$

$$\sqrt{40}$$

Does the other method make sense mathematically? Why or why not?

How are the methods similar? How are they different?

Engaging with Constructive and Non-Constructive Proof

Joe Garofalo, Christine P. Trinter, and Barbara A. Swartz

Conjecture and proof are the twin pillars of mathematics. . . . The concept of proof . . . brings something to mathematics that is missing from the other sciences . . .mathematicians have ways to build a logical argument that pins the label of "true" or "false" on practically any conjecture.

—Peterson 1988, pp. 217–18

One method of proof is to provide a logical argument that demonstrates the existence of a mathematical object (e.g., a number) that can be used to prove or disprove a conjecture or statement. Some such proofs result in the actual identification of such an object, whereas others just demonstrate that such an object exists. These types of proofs are often referred to as constructive and nonconstructive, respectively.

In this article, we share four tasks that we use to encourage secondary school students and preservice mathematics teachers to consider the conditions under which an example or counterexample, or even the logical demonstration that an example exists, can serve as a proof. We have regularly observed that students and others working through these tasks expand their approaches to proving statements and solving nonroutine mathematical problems.

The use of these tasks supports NCTM's Reasoning and Proof Standard for Grades 9–12, which includes recognizing reasoning and proof as fundamental aspects of mathematics; making and investigating mathematical conjectures; developing and evaluating mathematical arguments and proofs; and selecting and using various types of reasoning and methods of proof (NCTM 2000, p. 342). Also supported are aspects of the Number and Operations, Algebra, and Data Analysis and Probability, and Connections Standards. Moreover, we have found that solvers engaging in these tasks use several Standards for Mathematical Practice (SMPs) from the Common Core State Standards for Mathematics, particularly making sense of problems and persevering in solving them; reasoning abstractly and quantitatively; and constructing viable arguments and critiquing the reasoning of others (CCSSI 2010, pp. 6–8).

These four tasks can be classified into two types on the basis of how they can be resolved. Tasks 1 and 2 can be resolved by finding or constructing a specific example or counterexample that proves the given statement, whereas tasks 3 and 4 can be resolved by showing that an example or counterexample must exist, even if it is not constructed. We ask readers to try resolving these tasks before reading the solutions. Engaging with the tasks will give a better appreciation for the various solution strategies.

CONSTRUCTIVE-PROOF TASKS

Statements can be proved or disproved with an example.

Task 1: Batting Averages and Simpson's Paradox

This task may be stated as follows:

> Consider two baseball players, A and B. In the first half of the season, player A's batting average was higher than player B's batting average. During the second half of the season, player A's batting average was higher than player B's (again). Prove or disprove that for the entire season player B's batting average can be higher than that of player A. (Note: A batting average is calculated by dividing the number of hits by the number of at-bats; walks are excluded.)

This task involves a well-known statistical phenomenon, Simpson's paradox, which may not be novel to teachers but is not always introduced in secondary school mathematics curricula. The paradox sometimes arises when dealing with aggregate rate data or weighted averages, the latter of which is a standard topic in algebra courses and is usually addressed with mixture tasks, motion activities, and grade-point-average calculations.

Secondary school students are intrigued by the counterintuitive nature of this task. Most, at first, believe that it is not possible for player B to have the higher batting average, but many also feel that the task would be too easy if that were the case. Some solvers try to prove algebraically that it is not possible, often failing to consider the nature of batting averages. For example, **figure 1** shows the work of a preservice teacher who attempted to use a proof-by-contradiction argument but incorrectly summed ratios as fractions.

> "No, this is not possible. Let players A and B's batting averages for the first half of the season be denoted A_1 and B_1, respectively, and for the second half of the season, denote the averages as A_2 and B_2. So we know that $A_1 > B_1$ and $A_2 > B_2$. The problem asks, Is it possible for $B_1 + B_2 > A_1 + A_2$? Consider: $B_1 + B_2 > A_1 + A_2 > A_1 + B_2$, because $A_2 > B_2$, so $B_1 > A_1$. This is a contradiction; we know already that $B_1 < A_1$. So we have showed via contradiction that this is not possible."

Fig. 1 A preservice teacher's argument shows incorrect reasoning.

	A	B
1st half	.400 (2-5)	.390 (78-200)
2nd half	.360 (9-25)	.350 (70-200)
Whole Season	$.36\overline{6}$ (11-30)	.370 (148-400)

Fig. 2 A high school student offers a solution by example.

However, students who understand that players' batting averages are ratios search for at least one combination of hits and at-bats to prove that the scenario is possible. These solvers realize that the half-season batting averages cannot all be based on the same number of at-bats; thus some have more weight than others in the whole-season averages. By cleverly manipulating the number of hits in relation to at-bats, they find appropriate batting averages to show that player B can have the higher full-season batting average, as is seen in a student's solution using this approach (see fig. 2).

This high school junior's first reaction to the task was, "They can't have the same number of at-bats." When asked why not, she replied, "Sometimes if I went 6 for 10, a teammate would say she hit better by going 2 for 3 because it's a higher average . . . You really can't compare those because you don't always go 2 for 3."

When asked why she used 5 at-bats for batter A in the first half but 25 at-bats in the second half, this student explained, "I wanted to make [batter A's] first average go down a little. The 2 hits in 5 at-bats in the first half didn't do too much . . . They aren't strong enough compared to 25 at-bats." Then she added, "The 200 at-bats [for batter B] have more weight." From her experience as a softball player, this student had a feel for how weighting can affect overall batting averages, and she used this understanding in constructing her example. When debriefed about how she combined the batting averages for the season, she said, "This is not like adding fractions . . . you are adding totals for hits and totals for at bats, then dividing."

Figure 3 shows the work of a preservice teacher who first showed that "typical math" does not work here and then gave a combination of averages that solved the problem.

Students and preservice teachers provide and explain a range of solutions, and we ask them to describe common features of their solutions. This task generates class discussion pertaining to the meanings of fractions and ratios, the idea of weighted averages, and the nonintuitive context of the situation. This task supports NCTM's Number and Operations Standard, which states that students should be able to "judge the reasonableness of numerical computations and their results" (NCTM 2000, p. 393). As the student work suggests, judging the reasonableness of computations and results is critical to resolving this proof. Likewise, three Common Core SMPs (CCSSI 2010) are embodied in the students' perseverance in solving the problem, reasoning through the nonstandard" operations needed to calculate the batting averages, and their ability to justify their conclusions and communicate them to others.

" . . . in baseball batting averages, you do not add fractions as you would in typical math:

$$\text{Typical math: } \frac{1}{2}+\frac{2}{3}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$$

It doesn't make sense that someone could get 7 hits when he has only been up to bat 6 times.

$$\text{Baseball math: } \frac{1}{2}+\frac{2}{3}=\frac{3}{5}$$

In baseball math, you only need to add numerators and denominators straight across. Think of this logically. If [players] came to bat twice during the first half of the season and 3 times in the second half of the season, then they came up for a total of 5 times for the entire season. Using this notion, we can come up with a way to solve this problem.

	Player A		Player B
First Half	3 hits/9 at-bats	>	3 hits/10 at-bats
Second Half	1 hit/1 at-bat	>	7 hits/10 at-bats
Whole Season	4 hits/10 at-bats	<	10 hits/20 at-bats

. . . In the end, player A had a total batting average of .4, which is less than that for player B, who had a batting average of .5."

Fig. 3 A preservice teacher correctly uses an example of exist-

Note that Simpson's paradox can be observed with actual data, as shown in the batting statistics for Derek Jeter and David Justice during the 1995 and 1996 baseball seasons (Ross 2004, pp. 12–13) (see **table 1**). In both 1995 and 1996, Justice had a higher batting average than Jeter, but when the two baseball seasons are combined, Jeter has a higher average than Justice.

Table 1 Data from 1995 and 1996 MLB Seasons That Demonstrate Simpson's Paradox

	1995	1996	Combined
Derek Jeter	12/48 = .250	183/582 = .314	195/630 = .310
David Justice	104/411 = .253	45/140 = .321	149/551 = .270

Task 2: The Designing Dice Problem

Task 2 is worded as a question, but it can also be posed as a proof task. As a question, it can be answered by constructing an appropriate example. The task is stated as follows:

> If there are no restrictions on the numbers you can place on a pair of cube-shaped dice, is it possible to create a pair of dice such that you can roll all sums from 1 through 12—and only those sums—with equal probability?

We have given this task to middle school, high school, and undergraduate students and often get similar reactions. Some quickly respond that it is not possible to roll a sum of 1; these students have not kept in mind the condition "no restrictions on the numbers" on the dice and are still tied to the idea of standard dice. Others are stymied by the "equal probability" condition. Some initially try a few number combinations haphazardly, while others approach the task more systematically by resorting to their understanding of outcomes, sample spaces, and probability. These latter students realize that if there are 12 possible sums with 36 possible ways to get them, there must be 3 ways to get each sum.

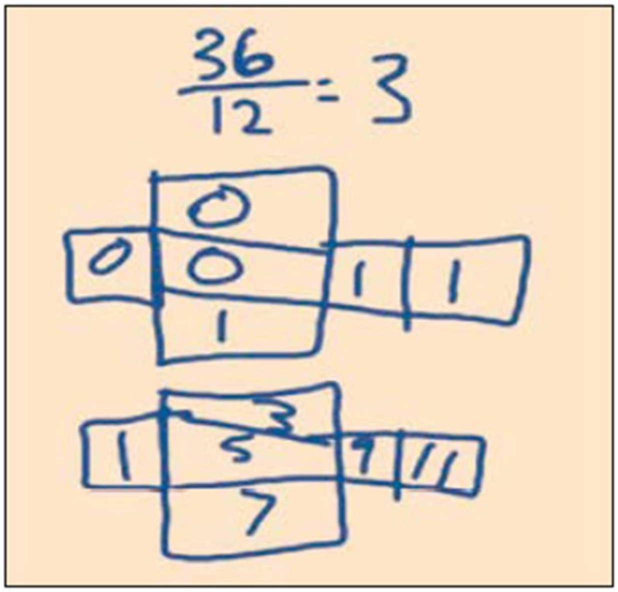


Fig. 4 One preservice teacher constructs a proof of the Dice task.

Figure 4 shows the work of a preservice teacher who used this thinking to construct an example. This teacher wrote the numbers as they might appear on a pair of dice. Most students write similar examples either in set notation, as the three shown in figure 5a, or in a chart, as the two shown in figure 5b.

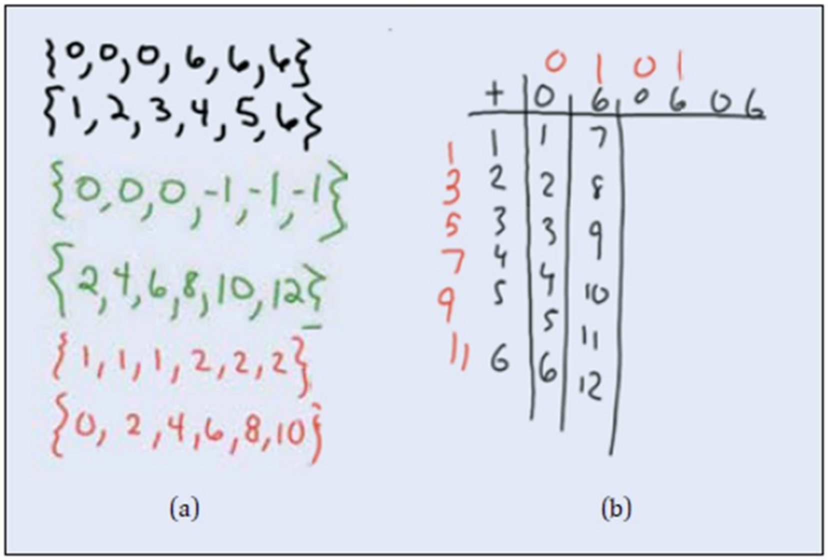


Fig. 5 Two students work to resolve the Dice task.

Although a few students initially do not think that creating such a pair of dice is possible, most of them eventually find at least one set of numbers that satisfy the task conditions and answer the question in the affirmative. Once several solutions are shared publicly, students see that all their solutions involve two numbers each appearing three times on a die, and they realize that there are an infinite number of possible dice that satisfy the conditions. Students connect solutions and observations to the definitions of probability and the notion of sample space, either while finding examples or after observing those found by others.

This task offers a unique way of addressing NCTM's Data Analysis and Probability Standard on two levels. Specifically, this Standard states that students should be able to "understand the concepts of sample space and probability distribution and construct sample spaces and distributions in simple cases" (the first level) and that students should "understand how to compute the probability of a compound event" (the second level) (NCTM 2000, p. 401). Further, when making connections among solutions, observations, and mathematical definitions, students "recognize and use connections among mathematical ideas" (NCTM 2000, p. 402), which is a cornerstone of the Connections Standard. This task also allows middle school students to "investigate chance processes and develop, use, and evaluate probability models" (CCSSI 2010, 7.SP.C, p. 50) while also engaging them in the SMPs.

NONCONSTRUCTIVE-PROOF TASKS

These statements can be proved by demonstrating that an example or counterexample exists.

Task 3: Prove or Disprove: An Irrational Number Raised to an Irrational Power Can Be Rational

At first glance, students often believe that this statement cannot be proven true. It is not until they consider familiar irrational numbers that solvers reconsider their initial reactions. This task helps students engage in number theory concepts in a unique way. One common pitfall for some students is failure to demonstrate or even state that the numbers they are using in their solutions are in fact irrational. This is a critical step in proving the statement and a significant practice in writing proofs. For example, the student whose work is shown in **figure 6** never even considered that his demonstration requires him to show that 1212 is itself irrational before raising it to an exponent.

Some solvers who are not completely certain about the irrationality of the number 1212 realize that they do not need to do so. If they can use this number to show that there exists at least one rational number that can be written as an irrational number raised to an irrational power, then the statement is proved. For example, the preservice teacher whose work is shown in figure 7 used the same numbers as the student whose work is shown in figure 6, but the logic was very different.

"At this point, I started looking at more properties of exponents and started looking at this one:

$$\left(a^b\right)^c = a^{bc}$$

I tried using the square root of 2 first and determined the following:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\bullet\sqrt{2}}$$

Here it is important to realize that

$$\sqrt{2}\bullet\sqrt{2} = 2.$$

Thus,

$$\sqrt{2}^{\sqrt{2}\bullet\sqrt{2}} = \sqrt{2}^{2} = 2.$$

Therefore, we see an example of when an irrational number raised to an irrational number is rational under one specific circumstance."

Fig. 6 A proof attempt demonstrates a common flaw.

This approach, shown in **figure 7**, considers two cases. Case A (which starts in the left column of work and ends with the top line of the right column) assumes that 12_{12} is a rational number and thus demonstrates the conjecture. Case B, on the other hand, assumes that 12_{12} is irrational. In this case, raising it to the irrational power 12 leads to

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = 2$$

Hence, case B also shows an irrational number raised to an irrational power equaling a rational number (i.e., the number 2). Only one of these cases must be correct, but the preservice teacher did not say which of the two is correct. However, doing so is not necessary to prove the given statement; this logical argument demonstrates that at least one example of an irrational number to an irrational power equaling a rational number exists. Because neither case and, hence, neither example is identified as the correct one, this proof is considered *nonconstructive*.

This task extends the Common Core Standards of Mathematical Content (SMCs) and concerns the meaning of rational exponents (i.e., HSN-RN.A.1) and understanding the sums and products of rational and irrational numbers (i.e., HSN-RN.B.3), by giving students an occasion to engage in the SMPs addressing problem solving, reasoning, and constructing and critiquing arguments (i.e., SMPs 1, 2, and 3). Moreover, the nature of this task gives students an opportunity to "develop an appreciation of mathematical justification" and requires that "their standards for accepting explanations should become more stringent" (NCTM 2000, p. 342). NCTM recommends that as students progress through high school, their level of sophistication with regard to proof increases.

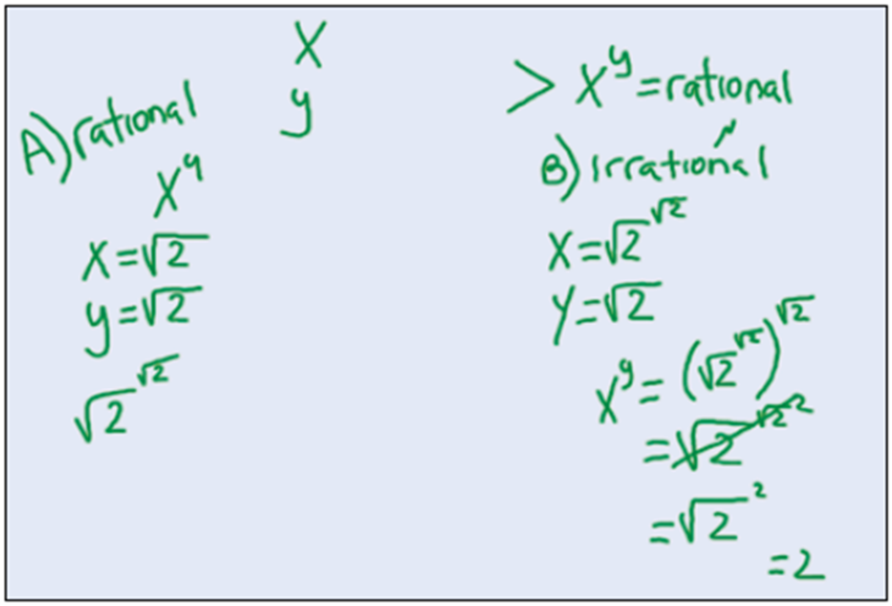


Fig. 7 A preservice teacher considers two cases in a nonconstructive proof.

Task 4: Prove or Disprove: In New York City, There Are at Least Two People with the Same Number of Hairs on Their Heads.

We usually present this task orally and often pause after the first three words, to which students respond with a combination of groans, sighs, and "Oh no." Then, after we get past a number of proposed trivial solutions, such as "Mr. Clean and Kojak," students usually start looking for information on the population of New York City and information on the average or typical number of hairs on human heads. Some use their found information to argue that there must be some people with the same number of hairs on their heads in New York City because the number of people in the city (e.g., 8,244,910) is so much larger than the typical number of hairs (or hair follicles) on heads (between 100,000 and 150,000). Most students respond that this is not a proof, but many of them also believe that it is a reasonable and almost convincing argument.

Other students use their found information to make a similar argument, but one based on the "pigeonhole" principle, which many have not formally learned in any mathematics course. For example, the high school student's work shown in **figure 8** suggests thinking consistent with an understanding of the pigeonhole principle.

This student assumed there are 8 million people in New York, found estimates for the typical number of hairs on a human head that ranged from 100,000 to 150,000, doubled 125,000 "to be sure," and reasoned that after one person was found with a hair count for each of the numbers from 0 to 250,000, there would still be 7,750,000 New Yorkers with a hair count equal to a number already used.

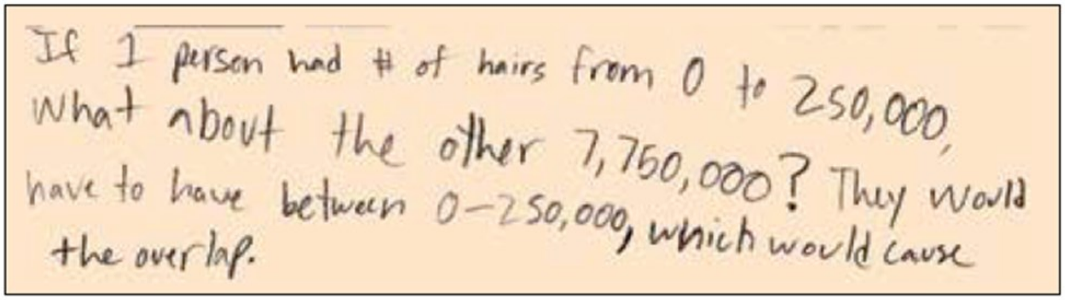


Fig. 8 A student's solution echoes the pigeonhole principle.

Similarly, **figure 9** shows the work of a preservice teacher who first stated that, on average, humans have 100,000 hairs but "to be safe assume [that] the maximum number of hairs would be 350,000 or even 1,000,000." She then argued that if one tried to sort the more than 8 million people in New York City into different "hair number" categories (pigeonholes), based on the number of hairs on their heads, after a million people were put into unique categories, the other 7 million would need to be placed in at least one of the already-taken categories.

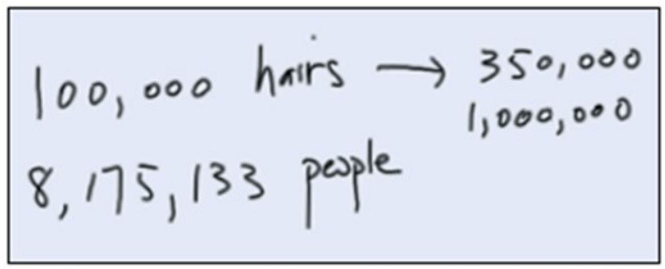


Fig. 9 A preservice teacher's work leads to a pigeonhole proof.

This is another example of a *nonconstructive proof*: Two people with the same number of hairs on their heads are not identified, but it is logically demonstrated that under the given conditions, they must exist. Indeed, a high school student remarked, "I know it's the case even though I didn't count the hairs on every person's head."

NCTM's Reasoning and Proof Standard states: "The repertoire of proof techniques that students understand and use should expand through the high school years" (NCTM 2000, p. 345). Task 4 provides students an opportunity to learn a proof strategy that may not be found in a typical high school textbook, a beneficial experience for them when conjecturing and justifying in unfamiliar mathematical situations.

PURPOSEFUL PRACTICE AND THE POWER OF PROOF

There are several points to keep in mind when using these tasks. Although they can be used to help students develop the ability to prove conjectures, if not used cautiously tasks can lead to misconceptions about proof by example. Teachers must be careful to help students see that an example does not and cannot prove many types of conjectures. Students need to thoroughly analyze statements and conjectures to be proved or disproved, carefully considering the meaning and significance of pivotal words such as *is*, *can*, *will be*, *always*, *all*, and so forth. Learning to judiciously analyze tasks like these will help students with other aspects of doing mathematics and other subjects as well.

Also, these tasks can be modified to meet teachers' specific goals and the needs of particular students. If you want your students to practice writing formal mathematical proofs, have them write up their arguments as such; however, if you want your students to practice mathematical critical thinking, ask them to brainstorm arguments and discuss them in pairs or small groups. In addition, present these tasks with various wordings, such as "Prove or disprove," "Is this possible?" "Explain when," or even "Find a case in which this works" (to simplify the tasks by restricting the outcomes).

Davis and Hersh argued that the purpose of generating a proof in mathematics has been for "validation and certification" (1981, p. 149) but added that a proof "increases understanding by revealing the heart of the matter . . . proof is mathematical power . . ." (p. 151). Indeed, the Common Core recognizes learning how to construct and evaluate proofs as an important component of a student's mathematical development:

> Mathematically proficient students . . . make conjectures and build a logical progression of state ments to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. . . . Mathematically profi cient students are also able to compare the effectiveness of two plausible arguments, distinguish cor rect logic or reasoning f rom that which is flawed, and—if there is a flaw in an argument—explain what it is. (CCSSI 2010, pp. 6–7)

Thoughtful use of the tasks presented here can help students develop mathematical power and proficiency.

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Middle School Practice #3 Activity

A "Justified List" is a strategy described in the book *Mathematics Formative Assessment: 75 Practical Strategies for Linking Assessment, Instruction and Learning* by Page Keely and Cheryl Rose Tobye. You begin with a collection of examples and non-examples focused around a statement or question. Students choose which examples fit the topic. For each example, students justify the rule, or reason for the selections. As students justify selections, this activity can be used to engage in math practice #3. Students are creating arguments as they complete the activity, and by sharing work with others, students may critique the reasoning of others. The "Justified List" topic can be changed to fit a variety of different content standards. This "Justified List" is designed for middle school students, focusing on the difference between linear and proportional relationships.

What is a Linear Relationship?

Circle all that represent linear relationships. Explain why each figure you circled represents a linear relationship.

A.)

1	2	3	4	5	6
5	7	9	1 1	1 3	1 5

B.)

1	2	3	4	5	6
3	6	9	1 2	1 5	1 8

C.)

1	2	3	4	5	6
1	4	9	1 6	2 5	3 6

D.)

0	1	2	3	4	5
0	3. 5	7	1 0. 5	1 4	1 7. 5

E.)

1	2	3	4	5	6
2 0	1 8	1 6	1 4	1 2	1 0

F.) F.) $5x = y$

G.) $5x + 1 = y$

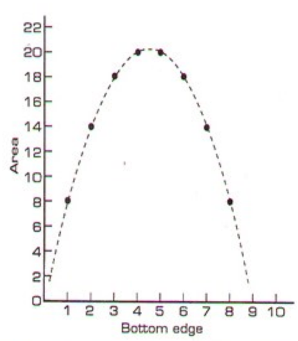
H.) $2.5x = y$

I.) $x^2 = y$

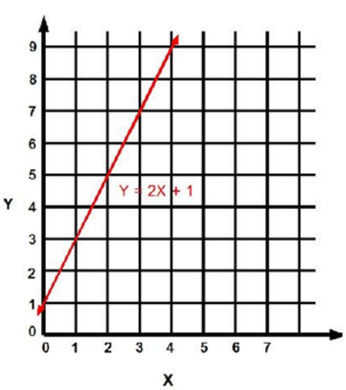
J.) Jamie gets paid $10.50 per hour. Show the relationship between the number of hours and her total pay.

K.) A phone costs $150 to buy, and $70 a month for service. Show the relationship between the number of months and the total cost.

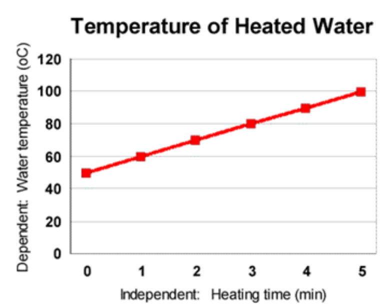
L.)



M.)



N.)



What is a Proportional Relationship?

Circle all that represent proportional relationships. Explain why each figure you circled represents a proportional relationship.

A.)

1	2	3	4	5	6
5	7	9	11	13	15

B.)

1	2	3	4	5	6
3	6	9	12	15	18

C.)

1	2	3	4	5	6
1	4	9	16	25	36

D.)

0	1	2	3	4	5
0	3.5	7	10.5	14	17.5

E.)

1	2	3	4	5	6
20	18	16	14	12	10

F.) F.) $5x = y$

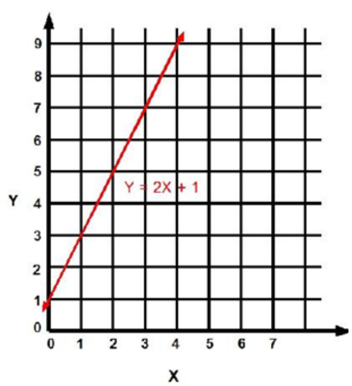
G.) $5x + 1 = y$

H.) $2.5x = y$

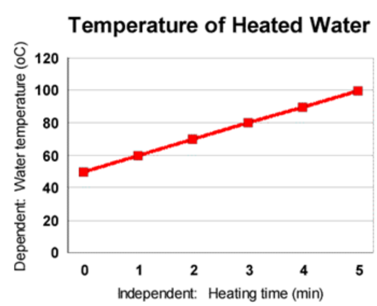
I.) $x^2 = y$

J.) Jamie gets paid $10.50 per hour. Show the relationship between the number of hours and her total pay.

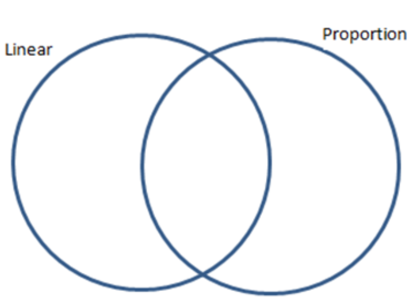
K.) A phone costs $150 to buy, and $70 a month for service. Show the relationship between the number of months and the total cost.

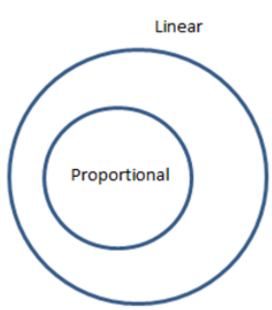


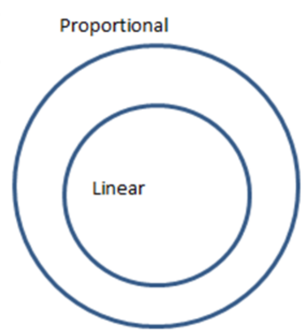
N.)



Which Venn diagram correctly shows the relationship between linear and proportional relationships? Justify your choice by using several examples from above. Compare and contrast linear and proportional relationships. Address similarities and differences that can be seen in tables, graph, equations and situations.







Establishing Standards for Mathematical Practice

Michelle L. Stephan

Have you ever tried to get a middle school student to explain her reasoning in front of her peers? In your attempts to have students understand one another's reasoning, how many times have you heard, "I don't get it! Any of it!" And how do your middle school students react when someone makes a mistake? Common Core State Standards for Mathematics (CCSSM) expects teachers to establish problem-solving environments in which students create their own solutions to problems, critique the reasoning of their peers, and come to a consensus on viable mathematical strategies and solutions (CCSSI 2010). However, these Standards for Mathematical Practice (SMP) are difficult for most middle school students to enact, and CCSSM gives no information on how to help students embody these practices. This article outlines some teaching strategies from my own middle school mathematics classes in an attempt to help my students implement the SMP.

One particular school year, I videotaped my eighth-grade class during the first two weeks of school. The purpose was to document the strategies that enabled students to problem solve and to discuss their solutions without fear of embarrassment. CCSSM was not yet adopted, so research on social norms guided the development of my teaching practice. Social norms are the expectations that the teacher and students have for each other regarding ways of participating in classroom discussions. These social norms, the most prominent in Standards-based classrooms, expect students to—

1. explain their reasoning to others;
2. indicate agreement or disagreement;
3. ask clarifying questions when they do not understand; and
4. attempt to understand the reasoning of others (Cobb et al. 1992; Stephan and Whitenack 2003).

These social norms align with many of the SMP, particularly the standards of making sense of problems and persevering in solving them (SMP 1), reasoning abstractly and quantitatively (SMP 2), and constructing viable arguments and critiquing the reasoning of others (SMP 3). The excerpts that follow contain details of some of the strategies I have developed to make these SMPs and social norms come alive in my middle school classroom.

SPECIFIC TECHNIQUES

The first two weeks of the school year were committed to setting social norms through general problem solving (SMP 1). For example, the problem that students encountered when they entered my classroom for the first time is shown in figure 1.

Fig. 1 Solving the Bacon problem, the first task of the school year, sets the stage for the forms of conversations that students can expect throughout the year.

> Drake found a sale on bacon at the Winn-Dixie. They were selling 4 packages for $10.
>
> - How much would Drake pay for 6 packages?
> - How much would he pay for 1 package?

At the beginning of the first day of class, the eighth graders were greeted as they walked in. They were then asked their names and asked to sit next to a person who they thought would be helpful during math class. They were also told to try the problem on the whiteboard. Because students had studied ratio and proportions in the seventh grade, I expected this problem to be accessible to all students, including those who were identified with special needs.

As students were problem solving, I monitored their work in a fashion similar to that described in 5 Practices for Orchestrating Mathematics Discussions (Smith and Stein 2011). During my monitoring time, students were encouraged to write their thinking on paper and to talk with a peer when they were stuck (SMP 1). I also gathered information on how students solved the problem so that I could select students who had solution processes that would be helpful for the follow-up class discussion.

Because of my agenda of establishing social norms, students presented their ideas in a specific sequence. For example, Brianna went first because her solution method was highly computational and would need deeper explanation for other students to understand. Her presentation would give the class the opportunity to discuss the importance of explaining and asking questions. The six strategies that follow guide the establishment of social norms for productive problem solving .

Strategy 1: State Expectations Early

That opening period would be the first whole-class discussion of the year, so it was critical to use it to begin the process of developing the norms that would drive the students for the rest of the year. Therefore, I engaged them in a discussion about ways to participate in the upcoming discussion. They were told that their job is to listen. When students were prompted to explain why, Anders responded, "You might like their ideas." To this statement, I added the following comments:

> You might like their ideas. And then you should steal it, right? Steal their ideas? That's why I want you to listen. I'm going to be asking you when somebody's up here presenting, Arthur, I want you to be listening, trying to understand t heir way. What if you didn't do it their way? Should you try to understand their way? Or just say, "Who cares? They didn't do it my way, I don't need to listen." No! You need to listen because as Anders just said, you might like their way better, right?

I also added this comment:

> I'm going to be asking you questions. Let's say that Brianna is up here explaining. I'm going to ask someone to explain what Brianna just did. And if you can't, you need to ask a question.

The expectation that students should listen to others' explanations was explicitly outlined. I asked them specifically what their "job" is when someone is explaining, then solicited reasons why this was important. However, if the SMP of analyzing and critiquing other students' arguments is going to become routine, stating these expectations upfront is not enough. Teachers must hold students accountable for listening, so I stated my accountability technique: I am going to call on students to explain what another student just argued.

Strategy 2: Hold Students Accountable for Explanations

To begin the whole-class discussion, Brianna came to the board to explain her approach to solving the Bacon problem in figure 1. At this prompt, Brianna realized I was expecting more than just an oral presentation from her desk. Our class conversation can be found in figure 2. I again called on the class to anticipate the rationale for certain expectations. Arthur argued that it is easier for the speaker just to read it, so I asked about the needs of the "audience." Anders explained that it would be easier for the audience to understand if Brianna wrote something, and she complied (see fig. 3).

As Brianna wrote her process on the board, I challenged the students to see if they could figure out her reasoning while she wrote it. This was my way of communicating to students that they should be analyzing Brianna's work and attempting to understand it (SMP 3).

Fig. 2 This conversation demonstrates how the explain-your-thinking expectation became the norm.

Brianna: Oh, I have to write it, too?
Teacher: Well, what do you guys think? Should she write something on the board, or should she just read it from her paper?
Arthur: It's easier to read it from your paper than write it.
Teacher: It *is* easier when you're the speaker. Right, Hugo? To just speak it from your paper? What about if you're the person listening? What's easier for you if you're trying to learn?
Anders: It's easier [for the listener] to read it.
Teacher: It's easier to read it, Anders says. So what do you think? What's your advice to her? Should she write something down for you to read?
Students: Yeah!

I paused for a few seconds of thinking silence and asked Brianna to move to the side so that everyone could analyze her written argument (SMP 3). The statements, "I think we're ready" and "Can you explain . . . to everybody," rather than "I think I am ready" or "Can you explain . . . to me?" are ways to let students know that the purpose of explaining is for the students to analyze and verify solutions, not just the teacher.

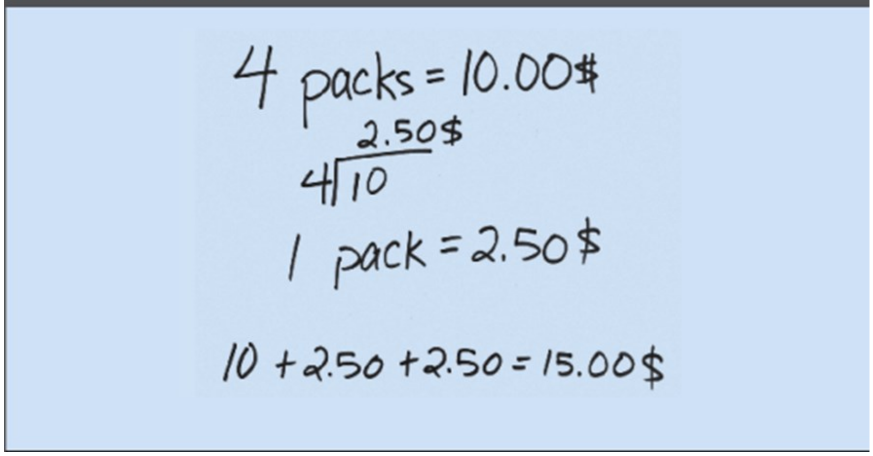


Fig. 3 At the request of classmates, Brianna demonstrated her reasoning by writing the explanation on the board.

As a result of this prompting, Brianna said:

> I knew that 4 packages was $10.00. So I did 4 divided by $10.00 and got $2.50, so one package was $2.50. And then I knew I had to get 6 packages so I added the four and one was $2.50 plus $2.50 was $5.00. And $10.00 plus $5.00 is $15.00.

Strategy 3: Hold Students Accountable for Asking Questions

The computational explanation given by Brianna did not contain an indication of what her steps and the results meant, so I searched the room to see what sense students had made of her work. I asked students to indicate whether they agreed or not by asking:

> Could you re-explain it? [They indicated yes.] Hold your hand up if you've got a question. Before she escapes [sits down], anybody got a question for her? That means if I call around, you'll know, you'll be able to explain her way, right? Alright. I'm going to call around. Valerie, you're her partner. What'd she do, how'd she solve this one?

Valerie responded, "I don't know." Because Brianna's explanation was very procedural (see Thompson et al. 1994), many students probably did not connect why she added $2.50 twice to $10.00. This procedural explanation gave me the opportunity to hold students accountable for asking questions when they did not understand. I let students know that I was about to ask others to re-explain Brianna's solution method. As expected, no one raised a hand, so I called on Valerie to reexplain. When she admitted she could not, I took this time to re-iterate my expectation that students should raise their hand if they do not understand. "Does anyone have a question?" is a weak way to hold students accountable because they can answer "yes" and the teacher moves on. The teacher should follow up and ask specific people to re-explain; if they cannot, reiterate the expectation.

Strategy 4: Hold Students Accountable for Making Sense of Solutions

Asking questions is important for helping students understand others' solutions, but the teacher must help students know what questions to ask. Most students give very calculational explanations (i.e., the steps that were taken) for their solutions (Thompson et al. 1994). The teacher's role is to push students toward more conceptual explanations in which the student explains why a particular calculation was made and what the results of that calculation mean in terms of the quantities in the problem situation (SMP 2). Thompson and others (1994) contend that conceptual explanations are more beneficial for struggling students because the reasons behind the steps are revealed to them .

To begin the sample discussion in figure 4a, Jamie simply restated the steps that he used to solve the problem. To press for understanding, I asked Judson what the numbers on the "top" stood for, so that students could draw connections to the quantities in the Bacon problem. As a result, Mariana brought out the term unit rate, and Marta related to students what that meant in this situation.

Strategy 5: Hold Students Accountable to Question What They Do Not Understand

Often when students do not understand the explanation of another student, the teacher presses them to ask a question. At the beginning of the school year, the most frequent response in my class is "I don't get it! Any of it." How does a teacher handle this exclamation without re-explaining everything in his or her own words? How can we teach students how to identify their areas of misunderstanding?

As the dialogue in figure 4a continued, Keisha responded that she really did not understand Jamie's solution. In the subsequent exchange with Keisha, shown in figure 5, I attempted to model how a student can go through the steps of the process and how to ask themselves, "Do I understand this part?" As it turned out, Keisha was able to explain almost all the steps except when misreading Jamie's writing as 250 rather than 2.50. Such a misinterpretation caused her difficulty in making sense of the rest of the solution. Helping to find the misunderstanding was an effective strategy when students indicated that they did not understand an entire solution. With enough of these explicit conversations, students began to analyze others' solutions to find specific places about which they could ask questions rather than say they misunderstood the entire solution.

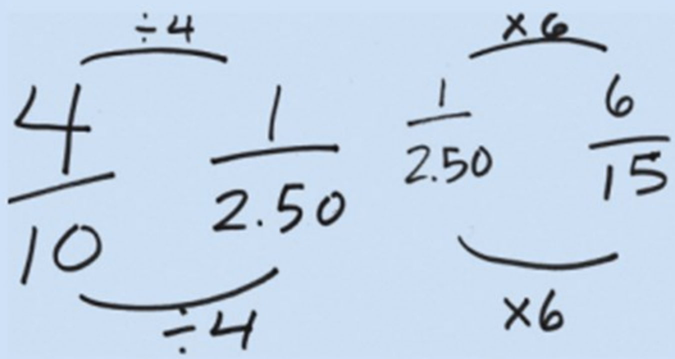
CCSSM Practices in Action

SMP 1: Make sense of problems and persevere in solving them.
SMP 2: Reason abstractly and quantitatively.
SMP 3: Construct viable arguments and critique the reasoning of others.

Fig. 4 In this discussion, the teacher pushed the students to expound on the calculations.

Jamie: OK, so when I first looked at the problem, it said that 4 packages was \$10. So 4 and 10. And then I tried to get to 1. I divided 4 by 4 to get 1 and 10 by 4 to 2.50. Then I did 1 times 6 to get 6 and then 2.50 times 6 to get to 15 [see **fig. 4b**].
Teacher: Alright; Judson, what do these numbers on the top stand for in the problem, for Jamie? What are they standing for?
Judson: The number of packages.
Teacher: How many packages? Mariana, what were you gonna say?
Mariana: Wouldn't it be like finding the unit rate?
Teacher: Woo! Did you hear that? What do you think? That was kind of directed at you [class]? What did she just ask? What did Mariana just ask? [No one raises their hand.] Oh no! All listen. Mariana, ask again real loud.
Mariana: Wouldn't that be finding the unit rate?
Marta: That one package costs 2.50.
Teacher: That one package costs 2.50. There it is. And then you used that unit rate, Jamie, to find out how much 6 packages cost. What do you think about Jamie's way?

(a) Class discussion



(b) The calculations in Jamie's strategy

Fig. 5 In this excerpt, the teacher helped students push through and reach understanding.

Teacher: Oh, what part don't you get?
Keisha: The whole thing.
Teacher: Oh no! The whole thing! Let's start here. Do you understand that part?
Keisha: Uh, 4 over 10? That four, that four packages cost \$10.00.
Teacher: You understand it, that part! [with excitement] What's this part all about, Keisha? See if she can get it. I bet she can, don't you? What's that next part all about?
Keisha: Does it say 250?
Anders: That's 2 point 50.
Keisha: Oh! That one pack costs \$2.50.
Teacher: And then what? What's this stand for?
Keisha: That's one package, and it costs \$2.50.
Teacher: Uh-huh. What's this stand for?
Keisha: 6 packages and \$15.00?
Teacher: Uh-huh.

Strategy 6: Praise Students for Their Participation and for Providing Informative Feedback

To end the first discussion of the school year, it is extremely important to stop in a similar manner as started: being explicit about expectations for current and future participation. This helps students understand the characteristics of their participation that were acceptable in class and those that needed to be improved. In addition, it is important to point out which social norms and Standards for Mathematical Practice were not acted on very well. For example, at the close of the Bacon problem, students were praised for their willingness to present their arguments in front of their peers, acknowledging how nervous it might have made them. The class applauded each person who presented and congratulated students on listening and being able to re-explain students' arguments. I also stated that we had some work to do asking questions when students do not understand one another. However, we would begin working on that more during the next class period.

ESTABLISHING NORMS TAKES TIME

The objective of this article was to describe the strategies (see fig. 6) for establishing social norms that are consistent with CCSSM's Standards for Mathematical Practice. A classroom environment in which students persevere in solving problems and feel engaged and safe enough to explain their thinking to their peers can have a positive effect on their learning (Tarr et al. 2008). However, social norms for communicating productively in class do not arise fully formed on the first day of class. It takes weeks for the teacher and students to establish these norms and for them to become stable for the rest of the school year. For this reason, I do not attempt to establish every single norm on the first day.

I reserve the first two weeks for building strong social norms and SMPs. Two weeks is not only a reasonable time period but also essential for setting the stage for communication in later units that target specific content. I recommend using general mathematics problems that elicit a variety of strategies and are not focused on developing new knowledge in a particular domain. Doing so allows the teacher to focus his or her attention explicitly on establishing problem-solving norms rather than on developing students' knowledge of one particular mathematics concept.

Fig. 6 This listing describes strategies for establishing standards-based social norms in math class.

Strategy 1: State expectations before the first explanation occurs
The teacher—
- states his or her expectations explicitly before a whole-class discussion begins; and
- engages students in developing the rationale for some of the norms.

Strategy 2: Hold students accountable for explaining
The teacher—
- calls on as many students as possible and uses names on day one;
- calls on students by name to explain their reasoning; and
- expects students to explain both verbally and in written form while engaging students in understanding the rationale.

Strategy 3: Hold students accountable for asking questions
The teacher—
- calls on students *by name* to see if they have questions;
- asks students to re-explain a solution method; and
- reiterates that it is unacceptable to violate certain norms (e.g., not asking a question when they do not understand).

Strategy 4: Hold students accountable for making sense of solutions
The teacher—
- asks students to not only re-explain but also describe why the particular procedures were taken and what the results mean in terms of the situation.

Strategy 5: Hold students accountable to question what they do not understand
The teacher—
- does not accept students' claims that they do not understand the entire solution explanation and instead teaches them how to analyze the strategy to find the misunderstanding.

Strategy 6: Praise students for their participation and for providing informative feedback
The teacher—
- applauds students for meeting norm expectations and provides feedback when certain norms are not being met effectively.

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Getting Started on Math Practice #3 in middle school classrooms –

Liz Peyser, VP Middle Schools

The very nature of math education is changing with the implementation of our new KCCRS standards. Gone are the days of students quietly sitting working out rows of computation problems. With the inclusion of Math Practices, students are using manipulatives, developing models and communicating their reasoning. Working with partners or small groups on problems strengthens students' reading, speaking, writing and listening skills, as well as their math skills. Sharing "entry points", ways of thinking about how to tackle the problem, and methods/strategies used will enable students to combine Practices 1, 4 and 5 as they develop an explanation and argument in Math Practice #3, this bulletin's focus Practice.

Having students exhibit Math Practice #3 can be a bit disconcerting for classroom teachers who didn't always learn this way and might be apprehensive to start classroom discussions. Sometimes the teacher feels as though they themselves aren't sure of the explanation that the student is giving and requires the teacher to really think through the math, misconceptions and possible pitfalls before assigning the problem to the students. Here are some simple steps toward beginning to incorporate class discussions.

Start small and build up the comfort level of yourself and students:

Step 1: Create or assign a problem with multiple entry points and chances to use multiple representations (measuring tools, table, graph, equation, number line, manipulatives) and have students work in partners. If good tasks are lacking, you can create a higher-level (error-analysis) problem from a simple problem by providing the answer, correct or incorrect. Students have to justify if it is the correct answer or not. Always solve the task/problem yourself and record items that might need explanation or areas of misconception.

Step 2: Display these two questions and use them daily to move from "answer-getting" to explaining:

"How do you know?"
"How is ______ related to ______?"

Step 3: Always ask "the next question"….that follow-up question to push the student's thinking.

For example, a possible problem could be to find the line of reflection of a shape and its reflection. *"Here is a quadrilateral and its reflection. Draw the line of reflection. How do you know that is the line of reflection?"*

Students can use protractors, mirras, compasses, ruler, straightedge, folding paper and other tools. There will be a variety of ways to tackle this problem. Students should be able to verbalize that the line is equidistant from points on the quadrilateral and its reflection at right angles. The "next question" might be "How is the line of reflection related to the midpoint of each segment that connects the reflected points" (the line of reflection becomes the perpendicular bisector of these segments). "Next question: "what does it mean to be a perpendicular bisector?" Another "next question" might be "What happens if you fold the paper on the line of reflection? What do you notice about the line segments and angles of the two shapes?" (the line segments and angles are preserved). "How do you know? How could you prove it?"

As students become more comfortable in sharing their methods, teachers might explore having students write answers on large whiteboards or chart paper. A diagram, short sentence and a math model might be all you need to keep it simple. Have the partners share their solutions and keep the class focused on asking/answering the Two Questions. Add more open-ended questions as necessary.

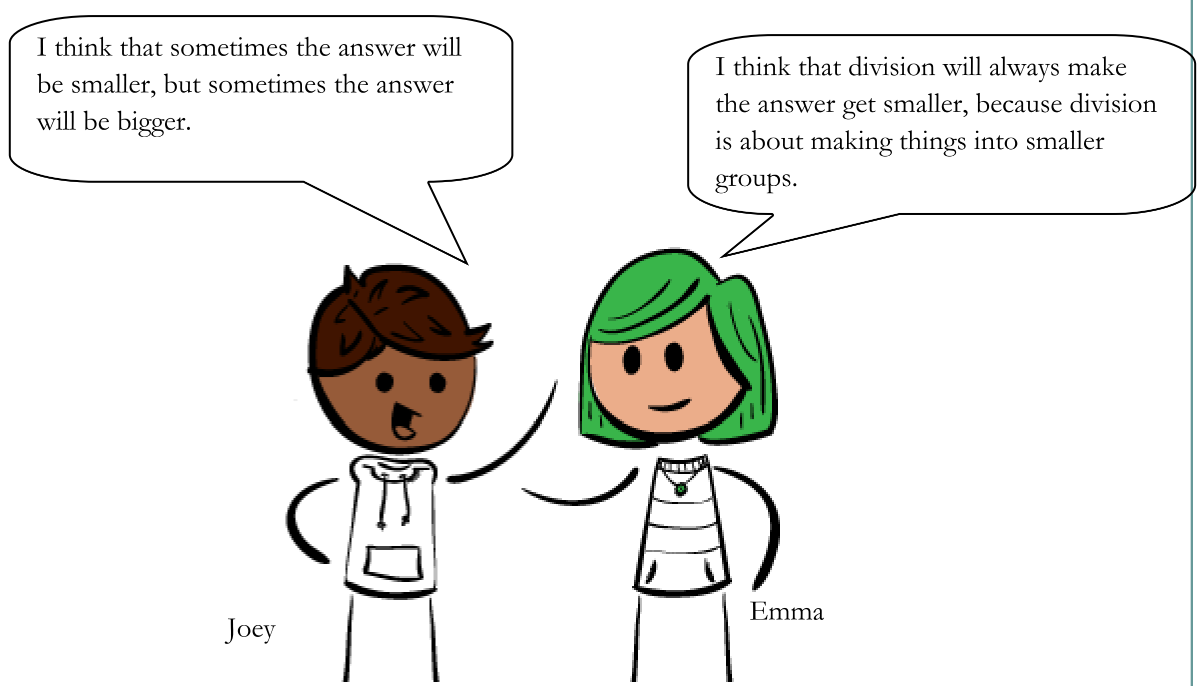
As teachers become more comfortable leading discussions, three valuable resources are *Classroom Discussions in Math* by Suzanne Chapin, et al. , *5 Practices for Orchestrating Productive Math Discussions* by Margaret Smith, and *Number Talks* by Sherry Parrish. These books provide "talk moves" and strategies for the teacher to employ to create an environment of math discourse. Tasks that promote student discourse can be located at Illustrativemathematics.org and map.mathshell.org.

Middle School Practice #3 Activity

Mathematics Formative Assessment: 75 Practical Strategies for Linking Assessment, Instruction and Learning by Page Keely and Cheryl Rose Tobye also describes the "Opposing Views Probe". In this strategy, students are presented with two or more different ways of thinking about a problem. It can be used to stimulate discussion and bring out student thinking. It encourages students to defend their viewpoint as well as listening to the viewpoints of others.

"Opposing Views Probes" are a great way to expose common student misconceptions and bring forth discussion about them. The activity below would be appropriate for an upper elementary/middle school classroom. This activity could be presented as a paper/pencil activity, where students justify their thinking in writing, or as a discussion tool. In a fun variation of this strategy, you can use pictures of famous people or characters that the students like giving opinions.

Division Makes Smaller



Which friend do you agree with? ____________________________________

Explain why you agree with one friend and disagree with the other. Provide evidence and/or examples that support your explanation.

NCTM – THE BENEFITS OF MEMBERSHIP

If you are not already enjoying the benefits of membership in the National Council of Teachers of Mathematics (NCTM), we invite you to learn more about membership and join today!

Some of the benefits of membership include:

- Print and online access to award-winning journals written specifically to your grade level: Teaching Children Mathematics (PK-6), Mathematics Teaching in the Middle School (5-9), Mathematics Teacher (9-12), Journal for Research in Mathematics Education, and Mathematics Teacher Educator
- Classroom-ready activities and materials with online teaching resources, plus a searchable database of challenge problems
- Discounts on hundreds of books, posters, and teaching materials
- Discounts on professional development opportunities: Annual Conferences, Regional Conferences, Interactive Institutes, and online seminars
- Grant opportunities exclusively for members: annual Mathematics Education Trust (MET) grants from $1200 to $24,000 are given to fund member project proposals

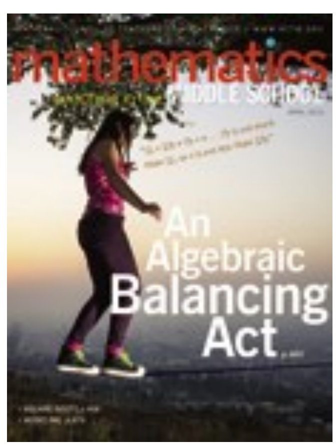
Support of a network of 80,000 mathematics educators from around the country and the world

Pre-service student members have several additional benefits including:

- Additional discounts on e-membership and add-on journals
- FREE registration to NCTM Regional Conferences

MET Grant opportunities exclusively for student members

Visit www.nctm.org/membership for more information on all of the benefits of NCTM membership.



KLFA

Advocacy in Action

April 16, 2015

Advocacy is who we are! Our April meeting began at the Kansas State Board of Education's (KSBE) April meeting, with many KLFA members present. During the Citizen Open Forum, KLFA members advocated for actions that support our goals: student success, professional learning, and community engagement.

After the KSBE meeting we regrouped at KASB and continued our collaboration. **Kelly Staton, KSDE Education Program Specialist,** led our "Learning" session by focusing on the new state accreditation model, **Kansas Education Systems Accreditation (KESA**). She shared the framework (5Rs: Relationships, Relevance, Responsive Culture, Rigor, and Results), timeline for districts, Outside Validation Team, similarities with QPA, and possible strategies/protocols that can be used with our organizations and local districts as we introduce KESA. Members were encouraged to make connections with other initiatives being implemented in their districts.

As part of a year long study of **change** and the **Standards for Professional Learning**, **Dayna Richardson**, KLFA chair, focused on the standard Learning Designs, as demonstrated by Kelly Slaton, focusing on multiple learning designs. The design depends on the purpose; one size does not fit all.

Finally, members were updated on legislative and policy issues, including ESEA. All members were encouraged to continue advocating for our students, our educators and our Kansas schools. The next meeting will be **June 11 at KNEA**.

For more information on KLFA, visit its Website at KLFA.org.

KATM Cecile Beougher Scholarship
ONLY FOR ELEMENTARY TEACHERS!!

A scholarship in memory of Cecile Beougher will to be awarded to a practicing Kansas elementary (K-6) teacher for professional development in mathematics, mathematics education, and/or mathematics materials needed in the classroom. This could include attendance at a local, regional, national, state, or online conference/workshop; enrollment fees for course work, and/or math related classroom materials/supplies.

The value of the scholarship upon selection is up to $1000:

- To defray the costs of registration fees, substitute costs, tuition, books etc.,
- For reimbursement of purchase of mathematics materials/supplies for the classroom

An itemized request for funds is required. (for clarity)

REQUIREMENTS:

The successful candidate will meet the following criteria:

- Have a continuing contract for the next school year as a practicing Kansas elementary (K-6) teacher.
- Current member of KATM (if you are not a member, you may join by going to www.katm.org. The cost of a one-year membership is $15)

APPLICATION:

To be considered for this scholarship, the applicant needs to submit the following no later than June 1 of the current year:

1. A letter from the applicant addressing the following: a reflection on how the conference, workshop, or course will help your teaching, being specific about the when and what of the session, and how you plan to promote mathematics in the future.
2. Two letters of recommendation/support (one from an administrator and one from a colleague).
3. A budget outline of how the scholarship money will be spent.

Notification of status of the scholarship will be made by July 15 of the current year. Please plan to attend the KATM annual conference to receive your scholarship. Also, please plan to participate in the conference.

SUBMIT MATERIALS TO:

Betsy Wiens
2201 SE 53rd Street
Topeka, Kansas 66609

Go to www.katm.org for more guidance on this scholarship

Capitol Federal Mathematics Teaching Enhancement Scholarship

Capitol Federal Savings and the Kansas Association of Teachers of Mathematics (KATM) have established a scholarship to be awarded to a practicing Kansas (K-12) teacher for the best mathematics teaching enhancement proposal. The scholarship is $1000 to be awarded at the annual KATM conference. The scholarship is competitive with the winning proposal determined by the Executive Council of KATM.

PROPOSAL GUIDELINES:

The winning proposal will be the best plan submitted involving the enhancement of mathematics teaching. Proposals may include, but are not limited to, continuing mathematics education, conference or workshop attendance, or any other improvement of mathematics teaching opportunity. The 1-2 page typed proposal should include

- A complete description of the mathematics teaching opportunity you plan to embark upon.
- An outline of how the funds will be used.

An explanation of how this opportunity will enhance your teaching of mathematics.

REQUIREMENTS:

The successful applicant will meet the following criteria:

- Have a continuing contract for the next school year in a Kansas school.
- Teach mathematics during the current year.

Be present to accept the award at the annual KATM Conference.

APPLICATION:

To be considered for this scholarship, the applicant needs to submit the following no later than **June 1 of the current year.**

- A 1-2 page proposal as described above.

Two letters of recommendation, one from an administrator and one from a teaching colleague.

PLEASE SUBMIT MATERIALS TO:

Betsy Wiens, Phone: (785) 862-9433, 2201 SE 53rd Street, Topeka, Kansas, 66609

KATM Election Results

Congratulations to our winners!

David Fernkopf, President-Elect

My name is David Fernkopf and I have been a member of KATM for over six years. I am currently the KATM Treasurer and was Zone 2 rep for five years. I am an elementary principal at Overbrook Attendance Center part of Santa Fe Trail USD 434. I feel that the work KATM does for teachers across the state is very important, and I would like to help with this work, by being a future leader for KATM.

Liz Peyser, VP Middle School

Middle school is squarely in the middle and the new standards have placed an increased importance on 6-8 learning! Middle school teachers need to know it all - what the K-8 standards are doing to prepare students and where the 9-12 standards take them. I have been involved in all aspects during the transition to new standards and will continue to provide guidance to middle school math teachers as Vice President of Middle Schools.

As a curriculum coach I research current issues around middle school math and offer responses at the district level. I have also worked closely with KSDE as a trainer-of-trainers for the KSDE math academies to provide professional development for the math standards to Kansas school districts. As the current Vice Principal for Middle School I bring this expertise to KATM to communicate information to all middle school math teachers across the state. I have worked closely with the editor to enhance the quarterly bulletin for readers. Our goal is for the bulletin to provide guidance in teaching math concepts and for understanding the Math Practices. This year each issue has a Math Practice focus, with all articles supporting understanding of that Practice. Using research, I contribute articles to this quarterly journal that focus on conceptual understanding of mathematics and how the standards work in learning sequences from K-8 or 6-12. I have also provided leadership around the contentious "acceleration" issue by providing an explanation and solutions to how the new standards will impact current middle school practices. With your support, I hope to be able to continue this work on behalf of Kansas math teachers.

Jerry Braun, VP College

My name is Jerry Braun. I am running for the position of Vice President for Colleges on the KATM Board. Mathematics education has been my passion for many years. I received my BA in Mathematics Education from Fort Hays State University in 1995 and my MS in Instructional Technology from Fort Hays State University in 2007. I have 15 years experience as a middle school mathematics teacher, 2 years experience as an Education Consultant for Southwest Plains Regional Service Center, 3 years experience as a k-12 instructional math coach and 7 years experience teaching online education and math courses for Fort Hays State University as an adjunct instructor. I have also filed to run for a position on the USD489 Board of Education. I have been a previous KATM board member serving as Zone 1 Representative, Vice President for Middle School and KATM President. Through this affiliation with KATM, I also served on the Common Core Standards Review committee, served twice as KATM conference chair/co-chair, and presented at numerous conference, KSDE summer academies and other training opportunities. I would like to rejoin the KATM board to help support mathematics instruction and preparation of mathematics teachers as well as math education in general. I look forward to hopefully serving you as a member of the KATM Board.

In the coming issues—

- *October 2015 Bulletin will focus on #6,* ***Attend to precision——***Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.
- December 2015—Make sense of problems and persevere in solving them
- February 2016—Reason abstractly and quantitatively
- April 2016—Look for and make use of structure AND Look for and express regularity in reasoning

CALL FOR SUBMISSIONS

Your chance to publish and share your best ideas!

The KATM Bulletin needs submissions from K-12 teachers highlighting the mathematical practices listed above. Submissions could be any of the following:

◊ Lesson plans
◊ Classroom management tips
◊ Books reviews
◊ Classroom games
◊ Reviews of recently adopted resources
◊ Good problems for classroom use
◊

Email your submissions to our Bulletin editor: wilcojen@usd437.net

Acceptable formats for submissions: Microsoft Word document, Google doc, or PDF.

Do you like what you find in this Bulletin? Would you like to receive more Bulletins, as well as other benefits?

Consider becoming a member of KATM.

For just $15 a year, you can become a member of KATM and have the Bulletin e-mailed to you as soon as it becomes available. KATM publishes 4 Bulletins a year. In addition, as a KATM member, you can apply for two different $1000 scholarship.

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Join us today!!! Complete the form below and send it with your check payable to

KATM to:
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Are you a member of NCTM? Yes___ No___

Position: (Cirlce only one)

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Referred by: ______________________________

____

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Three Years: $40 ___
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Institutional Membership: $25/yr. ___
Retired Teacher Membership: $ 5/yr. ___
First Year Teacher Membership:$5/yr. ___
Spousal Membership: $ 5/yr. ___
(open to spouses of current members who hold a regular Individual Membership in KATM)

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