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Kansas Association of Teachers of Mathematics

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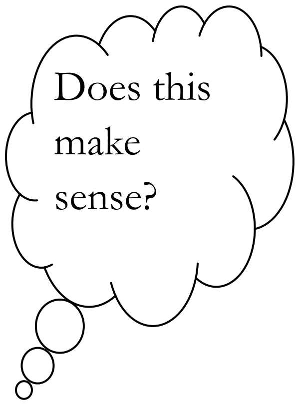
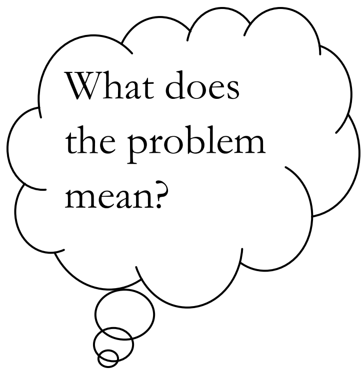




Make sense of problems and persevere in solving them

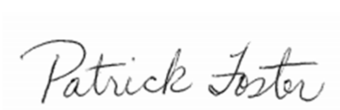
Inside this issue:

A Message from our President

Hello Kansas mathematicians! In this issue of the KATM newsletter, we are highlighting the first mathematical practice, which is making sense of problems and persevering in solving them. When students persevere, they are able to explain the meaning of the problem and look for multiple methods to approach and solve the problem. They continually ask themselves, "Does this make sense?" They might use drawings or concrete objects to help get a mental picture of the problem. Students are able to make connections to similar problems and plan a solution in a systematic manner. Finally, they are able to change course in the middle of a problem and approach it from another angle if their method is not working.

One strategy to help students makes sense of problems and persevere in solving them is to implement number talks in the classroom. This strategy can be used as a daily warm up or as part of an extended lesson. Students participate in discussions about various ways to solve challenging math problems. They struggle with interesting math concepts and talk about their particular method for solving the problem. While this strategy will help with fluency, the focus is not always on finding the right answer. Instead, the discussion and the ability to explain their mathematical thinking is key for the students. Students will then be able to apply this thinking to other problems in the future. And, they are taught that understanding a problem and taking time to work through it can be a valuable tool for learning mathematics.

If students are to be successful at solving complex problems, they must be able to communicate what a problem means and understand what they are being asked to do. Providing direct instruction using productive math talk can help facilitate this process for math learners. Consider joining KATM to network with other math professionals to help increase your knowledge of the Standards for Mathematical Practice and how to implement them in the classroom.



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In the coming issues—

February 2016—Reason abstractly and quantitatively

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

April 2016—Look for and make use of structure AND Look for and express regularity in reasoning

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Dear Kansas Math Teachers,

I hope you enjoy this latest issue in our series about the standards for mathematical practice. We started this series last year, and will finish it up at the end of this school year. The members of the KATM board strive to provide valuable resources to our members, and our Bulletin is one of the primary resources we have. What would you like to see next? We would love to hear from you with ideas about what you'd like to see in future Bulletins!

Jenny Wilcox KATM Bulletin Editor

Making Sense of Word Problems

By Lynette Sharlow, KATM Vice President Elementary

Math Practice 1 is an overarching skill to all other math practices and all math content. This article is going to narrow the focus to making sense of word problems. Making sense of various situations and knowing the type of situation the problem is presenting is a process that begins at the beginning of math instruction. It gives purpose to why we learn the skills we are taught in math. It is important to know that there are 21 different types of situations students need to know how to make sense of and solve. By the end of second grade, all 15 addition/subtraction situations must be able to be solved in two steps. Third and fourth grades make sense of the 9 multiplication/division subtypes and apply all twenty-one situations in multiple steps. (see tables below).

Common Addition and Subtraction Situations (pg 88 in CCSS)

Shading taken from OA progression

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Taken from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3
	Total Unknown	**Addend Unknown**	**Both Addends Unknown**[1]
Put Together/ Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare[3]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

Blue shading indicates the four Kindergarten problem subtypes. Students in grades 1 and 2 work with all subtypes and variants (blue and green). Yellow indicates problems that are the difficult four problem subtypes or variants that students in Grade 1 work with but do not need to master until Grade 2.

[illegible]

Common Multiplication and Division Situations (pg 89 in CCSS)

Grade level identification of introduction of problems taken from OA progression

	Unknown Product	Group Size Unknown ("How many in each group?" Division)	Number of Groups Unknown ("How many groups?" Division)
	3 x 6 = ?	3 x ? = 18; 18 ÷ 3 = ?	? x 6 = 18; 18 ÷ 6 = ?
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays[4]**, Area**[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	a × b = ?	a × ? = p, and p ÷ a = ?	? × b = p, and p ÷ b = ?

Multiplicative compare problems appear first in Grade 4 (green), with whole number values and with the "times as much" language from the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit language change the subject of the comparing sentence ("A red hat costs n times as much as the blue hat" results in the same comparison as "A blue hat is 1/n times as much as the red hat" but has a different subject.)

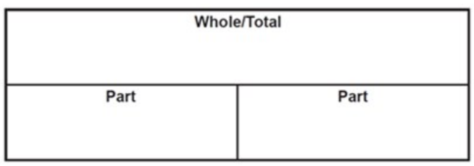
To help students make sense of what is given and what unknown needs to be solved for can be a challenge to everyone's need to persevere. We've recognized that the "key word" angle doesn't work so well once the total/result is no longer the "unknown" of the problem and two-step problems begin to happen. There are some "thinking tools" we've used in our district that help students break down the details provided in a problem and determine next steps for solving them. The "Start-Change-Result: What's the Unknown?" mat works best with problems that have action or a recognizable sequence. The action of the problem helps to determine if it's an addition, subtraction, multiplication or division event. Depending upon what the "unknown" is, students may need to depend on their understanding of the addition-subtraction or multiplication-division relationship to solve. For instance, if I don't what I started with but know that I lost 7 items and now only have 3 left, I would have to add to find my unknown start. Losing something is a subtraction situation, but I must depend on my addition-subtraction relationship understanding to solve.

"What's the Unknown?"

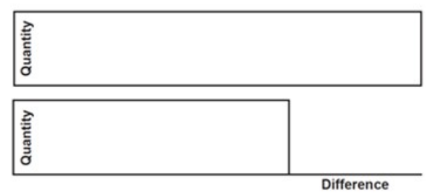
Start	Change	Result

Other tools that help make sense of the addition-subtraction relationship are the Part-Part-Whole mat, and the Comparison Model mat.

Part–Part–Whole/Total

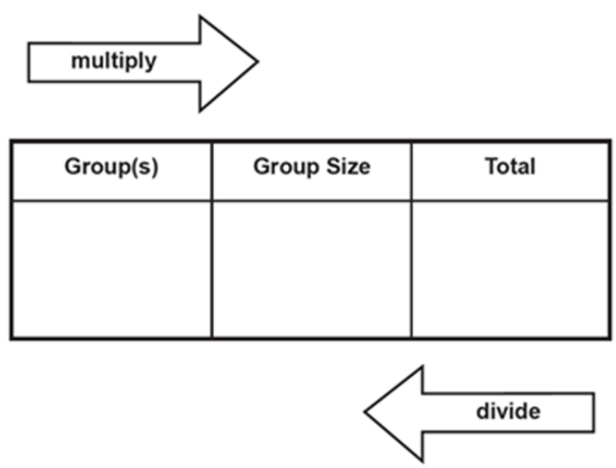


Comparison Model



The Comparison Model especially helps with understanding that subtraction doesn't always mean that I "take away". If I want to find the difference (the amount it would take to make both parts equal), then this could be determined to be a subtraction situation too.

The tool we use most to make sense of multiplication-division situations is the Group-Group Size-Total mat. We've found that the deeper the students' understanding of this relationship, the better the understanding students have when multiplying and dividing situations with fractions in the upper grades.



Keep in mind that instruction doesn't always have to go to the "number answer" for the problem. Sometimes what needs to be the greatest focus is knowing what is happening in the situation presented. Help empower students with the skills to determine a clear understanding of problems they are presented and the confidence to attack them.

The Richness of Children's Fraction Strategies

How would you decide when to give your students a problem that involves division of fractions, such as this Hot Chocolate problem:

> You have 4 2/3 cups of hot chocolate powder. Each serving requires 2/3 cup hot chocolate powder. How many servings can you make?

What kinds of strategies do you anticipate they would use, and what do you think they could learn?

In this article, we discuss a special type of multiplication-and-division-of-fractions problem that elementary school teachers can use to promote children's understanding of fractional quantities and their relationships. These problems are accessible to students working at different levels of fraction understanding, and they can be solved without the use of standard algorithms for multiplying and dividing fractions. Encouraging children to model the quantities and relationships in the problem situation helps them build a strong foundation for understanding fractions (Empson and Levi 2011; Warrington 1997).

This special type of problem includes equal-groups situations with a whole number of groups and a fractional amount in each group, such as the Hot Chocolate problem (Empson and Levi 2011). We call these multiple groups problems. In this article, we discuss how multiple groups problems can be used to promote the development of children's understanding of fractional quantities and their relationships before the introduction of generalized procedures for multiplying and dividing fractions. This type of story problem is appropriate for students of any grade level who are able to model fractional quantities and link these models to a problem situation.

Students' strategies for the Hot Chocolate problem

Children in the elementary grades can solve fraction story problems by drawing on their informal understanding of partitioned quantities and whole-number operations (Empson and Levi 2011; Mack 2001). Given the opportunity, children use this understanding to model fractional quantities, such as 1/4 of a quesadilla, and reason about relationships between these quantities, such as how much quesadilla there would be if 1/4 of a quesadilla, 1/4 of a quesadilla, and 1/4 of a quesadilla were combined.

To explore these ideas, we return to the Hot Chocolate problem and share four strategies generated by students in a fifth-grade class. When the classroom teacher, Mrs. Gardner, posed this problem, she deliberately did not show a strategy because she wanted to give students the opportunity to make sense of the problem on the basis of their own understanding of fractions. To engage her students, she encouraged them to relate to the problem situation by imagining a time when they made hot chocolate using a mix. Students shared their experiences, and Gardner helped them visualize a container of powder and the process of making a serving. Her goals were to support her students in making sense of the problem and generating a strategy that they could explain and justify.

Direct modeling of fractional quantities

Gardner's fifth graders used a variety of strategies that revealed a range of understanding of fractional quantities and their relationships. We present four strategies to illustrate this range, which is typical of what teachers can expect third, fourth, and fi fth graders to do for problems like this one (Empson and Levi 2011).

Maggi solved the Hot Chocolate problem by directly modeling the quantities in the situation (see fig. 1). *Direct modeling* describes student-generated strategies that involve representing all the quantities in a way that reflects the structure of the problem situation (Carpenter et al. 2015). Maggi represented each cup with a circle. She then partitioned each cup into thirds and shaded the last third in the fifth cup to show that it was it was not part of the 4 2/3 cups. To determine how many servings could be made by 4 2/3 cups of powder, Maggi numbered the thirds to group them into servings of 2/3 cup each. Thus, "1, 1" in her drawing represents the first serving, "2, 2" the second, and so on, up to "7, 7" for the seventh serving.

FIGURE 1

Maggi used a direct-modeling strategy for solving the Hot Chocolate problem.

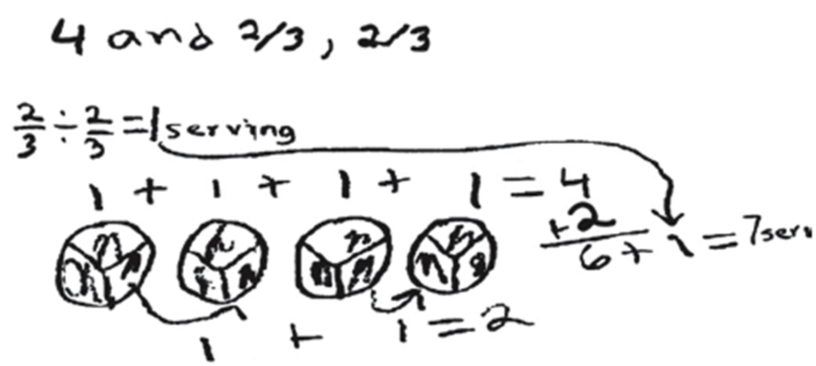
Maggi's strategy reveals her understanding of each cup as a whole unit and her implicit understanding that a serving of 2/3 cup can be decomposed into 1/3 cup and 1/3 cup. This understanding and her representation of the situation helped her determine that there would be seven servings of hot chocolate.

Gabriela also solved the Hot Chocolate problem by directly modeling the quantities (see fi g. 2). Additionally, she recognized right away that 2/3 cup from the quantity 4 2/3 cups would make one serving and wrote the number sentence 2/3 ÷ 2/3 = 1 to show her thinking. She then drew the remaining 4 cups of chocolate powder similarly to Maggi but counted the servings differently. First, she counted one serving from each of the 4 cups, as shown by her shading of 2/3 of each cup and the equation 1 + 1 + 1 + 1 = 4 above the cups. She then counted the final servings by combining 1/3 and 1/3 from each of 2 cups, twice, as shown by 1 + 1 = 2 below the cups. She added her counts to find a total of seven servings.

These direct-modeling strategies reflect children's early understanding of fractional quantities as a part of a whole and their understanding of division in terms of counting these quantities. This early understanding encompasses relationships that form a necessary foundation for more sophisticated understanding of fractions. Specifically, both Maggi and Gabriela had to understand and be able to reason about the quantity 2/3 cup as a combination of unit fraction quantities, 1/3 cup and 1/3 cup. They also had to be able to differentiate between what counted as a cup and what counted as a serving within these drawn quantities. Further, identifying a serving required some flexibility, in that a serving of 2/3 cup could be taken entirely from a single whole, or it could be composed of 1/3 cup and 1/3 cup from two different wholes of the same size. Engaging in direct-modeling strategies presents opportunities for students like Maggi and Gabriela to build their understanding of fractions as quantities and of the relationships between unit fractions and other fractions (Empson and Levi 2011).

FIGURE 2

Gabriela also used a direct-modeling strategy to solve the Hot Chocolate problem, but she counted servings differently.



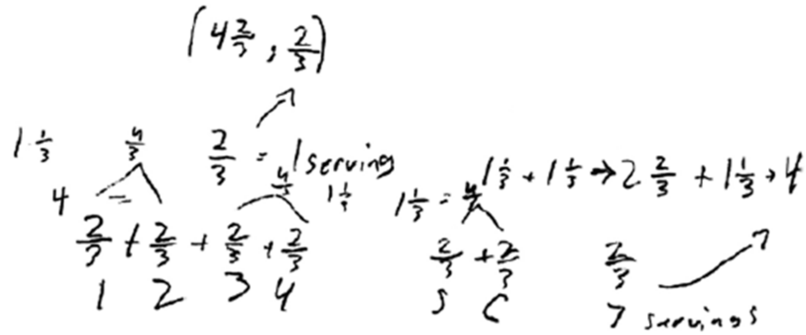
Using fraction relationships

Other students in Gardner's class approached the Hot Chocolate problem by relating fraction division to repeated addition or multiplication, using a more abstract understanding of fractions. These students did not need to represent the part and the whole as drawn quantities to reason about fraction relationships.

Jake solved the problem by using repeated addition (see fig. 3). He represented and reasoned about the fractional quantities symbolically. To successfully use addition to solve the problem, he had to keep track of when to stop adding groups of 2/3 and how many groups of 2/3 he had added. He started by adding four groups of 2/3, using a doubling strategy: 2/3 + 2/3 + 2/3 + 2/3 = 1 1/3 + 1 1/3. His addition reveals his understanding of relationships between fractions and whole numbers. For example, he added 2/3 and 2/3 and got 4/3, which he expressed as 1 1/3, because he knew that a whole cup consists of three groups of 1/3 cup. Using mixed numbers in this way helped him find the sum by adding whole numbers and fractions separately: 1 1/3 + 1 1/3 = 2 2/3. He saw at this point that he could add another grouping of 1 1/3 to get to 4 cups of chocolate powder: 2 2/3 + 1 1/3 = 4. Finally, he indicated that the remaining 2/3 cup of chocolate powder could make one more serving. Jake used addition to solve the problem, but he did not use a standard algorithm for adding fractions. Instead, he used what he knew about fractional quantities and their relationships to produce a mixed-number amount. This strategy provided an opportunity for Jake to work on combining fractions and to reason about relationships between fractions and mixed numbers. When he shared his strategy with the rest of the class, he showed how he numbered each group of 2/3 to keep track of how many servings there were as he added the fractional amounts.

FIGURE 3

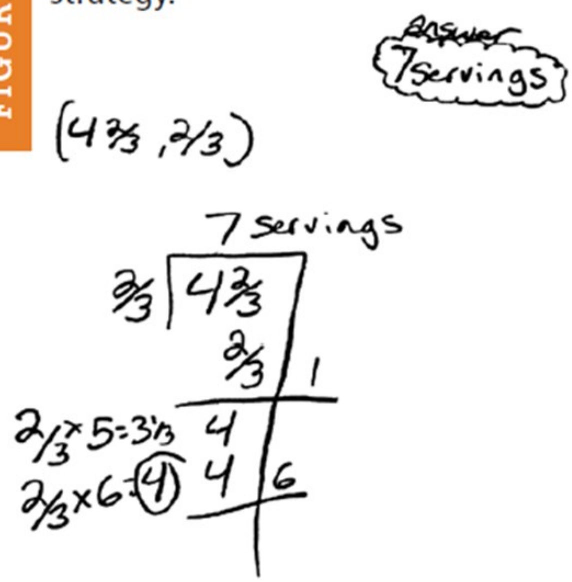
Jake's strategy was repeated addition.



Olivia related the division to multiplication (see fig. 4), by estimating how many groups of 2/3 cup would make 4 2/3 cups. Like many students in Gardner's class, she thought about 4 2/3 cups as 4 + 2/3 and realized that one serving would be possible with the 2/3 cup. Because she did not know how many servings of chocolate powder were needed for the remaining 4 cups, she estimated. She reasoned that she would be able to make at least five servings, because each cup could make at least one serving, with some thirds left. She multiplied 2/3 by 5 to get 3 1/3. She explained, "Five groups of 2/3 is the same as ten groups of 1/3. I know 1/3 × 9 is 3, and then 1/3 more would be 3 1/3." When she saw that this amount was 2/3 away from 4, she realized that she needed six groups of 2/3, rather than five groups, and that 2/3 × 6 = 4. Olivia's strategy reveals an understanding of the inverse relationship between multiplication and division. Rather than compute 4 2/3 ÷ 2/3 = c, she solved the problem by thinking of 2/3 × c = 4 2/3. She used a partial-quotients notation to organize her approach to solving the problem. Although this notation is usually reserved for multi-digit, whole-number division problems, Olivia used it in a novel way to notate her strategy with fractions, suggesting that she understands fractions as quantities on which she can operate just as she operates on whole numbers.

FIGURE 4

Olivia used the partial-quotients strategy.



Reflecting on students' strategies

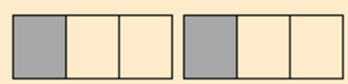
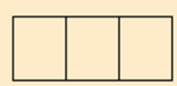
The strategies that students used allowed Gardner to address a number of mathematical content and practice goals. After each student had a chance to solve the problem individually, Gardner organized a discussion of selected strategies to allow students to reflect on one anothers' strategies and extend their understanding (Kazemi and Hintz 2014).

For example, as Maggi shared her direct modeling strategy and Jake shared his repeated addition strategy, Gardner asked the class to find Jake's sum of 2/3 + 2/3 in Maggi's strategy. She also asked how Maggi's representation of seven groups could be found in Jake's strategy. This discussion helped Maggi see how the quantities she had directly modeled could be symbolized, and it helped Jake communicate how his addition strategy worked. When Olivia shared her partial quotients strategy, the class discussed how operations on fractions and whole numbers are connected. Table 1 lists some of the fraction content evidenced in the children's strategies for the Hot Chocolate problem. This content reflects the children's developing understanding of fractions as quantities and their use of fundamental fraction relationships.

Further, as students engaged in solving and discussing the hot chocolate problem, they used a number of the Common Core's Standards for Mathematical Practice (SMP) (CCSSI 2010, pp. 6–7). For example, because Gardner did not demonstrate a strategy to use to solve the problem, students had to make sense of the problem and persevere in solving it (SMP 1). She chose a context—mixing hot chocolate—that elicited students' quantitative reasoning. For some children, such as Maggi and Gabriela, this reasoning was connected to models. For others, such as Olivia and Jake, it was abstract. Olivia's and Jake's strategies also made use of structural aspects of fraction operations, such as when Olivia related 5 × 2/3 to 10 × 1/3 and when Jake added mixed numbers by combining whole numbers first, then fractions. Engaging in practices such as these supported students in exploring and reflecting on their understanding of fraction quantities and their relationships.

TABLE 1

Students' strategies for the Hot Chocolate problem reflect their developing understanding of fractions as quantities and their use of fundamental fraction relationships.

Fractional quantities and their relationships	Possible representation
Model a fraction, such as 2/3 cup, as a drawn quantity.	
Two-thirds cup can be decomposed into one-third cup and one-third cup.	2/3 = 1/3 + 1/3
One cup is equal to three groups of one-third cup.	 1/3 1/3 1/3 1 = 3 × 1/3
A mixed number can be decomposed into a whole number and a fraction.	4 2/3 = 4 + 2/3
Multiplication is the inverse of division.	4 ÷ 2/3 = c 2/3 × c = 4

Promoting rich understanding of fractions

One of the most persistent difficulties with fractions is that students do not have enough of the kinds of experiences they need to make sense of fractions as quantities (Empson and Levi 2011; Hackenberg 2013). If students do not see fractions as quantities, they have difficulty making sense of operations on quantities, such as adding or multiplying.

Teachers can use multiple groups problems, such as the Hot Chocolate problem, to build children's understanding of fractions as quantities. Children do not have to be able to multiply or divide fractions to engage productively in solving multiple groups problems, as Maggi's, Gabriela's, and Jake's strategies, in particular, show. As long as children can create models of fractional quantities and explain how they relate to a situation, they are likely to be able to generate strategies for multiple groups problems.

Multiple groups problem types include measurement division and multiplication story problems (see table 2). In both cases, the problem situation has a whole number of groups, with a fractional quantity in each group. (The total amount in all the groups may be a whole number or a mixed number.) In a measurement division problem, the number of groups is unknown. In a multiplication problem, the total amount is unknown. The relative difficulty of a multiple groups problem can be adjusted by changing the number selection. For example, children tend to find it easier at first to solve problems that involve a unit-fraction quantity, such as 1/4 sub sandwich, as the amount in a group, and harder to solve problems that involve less familiar fractional quantities, such as 2/5 gallon of paint.

The strategies used by Gardner's students illustrate the richness of children's thinking about fractional quantities and their relationships. All students have the potential to generate strategies like these. If you are curious about your students' thinking, you might pose one of these problems to your class. Encourage students to imagine the situation and to visualize

TABLE 2

The sample multiple groups problems below are ordered according to their increasing difficulty.

Measurement division	Multiplication
Mom has 10 grilled cheese sandwiches. If she wants to give each child at the party a serving of 1/2 sandwich, how many children can have a serving of grilled cheese sandwich?	Dad wants to serve each child at the party 1/4 sub sandwich. If 8 children are coming to the party, how many sub sandwiches should he make?
It takes 3/4 cup sugar to make a batch of cookies. If the baker has 15 cups of sugar, how many batches of cookies can he make?	The baker wants to make 9 batches of cookies. Each batch uses 2/3 cup sugar. How much sugar does the baker need to make all the batches?
A carpenter is making dog houses that each requires 4/5 pint of paint. If he has 10 2/5 pints of paint altogether, how many dog houses can he paint?	A carpenter wants to make 12 dog houses. If each dog house requires 2/5 pint of paint, how much paint will the carpenter need?

the quantities and their relationships: Have they ever shared sandwiches or made a batch of cookies? Can they picture a pint of paint and painting a dog house? Allow students to follow their own line of reasoning, and then listen for what the strategies reveal about their understanding of fractions as quantities. The problem that Gardner selected could be solved in several different ways, and each student had a way to get started, from direct modeling of fractional quantities to combining fractions and reasoning about the inverse relationship between multiplication and division. Maggi, Gabriela, Jake, and Olivia all had something to contribute.

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KATM Cecile Beougher Scholarship—ELEMENTARY TEACHERS ONLY!!

A $1000 scholarship in memory of Cecile Beougher will to be awarded to a practicing Kansas elementary (K-6) teacher for professional development in mathematics, mathematics education, and/or mathematics materials needed in the classroom. This could include attendance at a local, regional, national, state, or online conference/workshop; enrollment fees for course work, and/or math related classroom materials/supplies.

- To defray the costs of registration fees, substitute costs, tuition, books etc.,
- For reimbursement of purchase of mathematics materials/supplies for the classroom

An itemized request for funds is required. (for clarity)

REQUIREMENTS:

- Have a continuing contract for the next school year as a practicing Kansas elementary (K-6) teacher.
- Current member of KATM (if you are not a member, you may join by going to www.katm.org. The cost of a one-year membership is $15)

APPLICATION:

To be considered for this scholarship, the applicant needs to submit the following no later than June 1 of the current year:

1. A letter from the applicant addressing the following: a reflection on how the conference, workshop, or course will help your teaching, being specific about the when and what of the session, and how you plan to promote mathematics in the future.
2. Two letters of recommendation/support (one from an administrator and one from a colleague).
3. A budget outline of how the scholarship money will be spent.

Notification of status of the scholarship will be made by July 15 of the current year. Please plan to attend and participate in the KATM annual conference to receive your scholarship.

SUBMIT MATERIALS TO: Betsy Wiens, 2201 SE 53rd Street, Topeka, Kansas 66609

Go to www.katm.org for more guidance on this scholarship

Capitol Federal Mathematics Teaching Enhancement Scholarship

Capitol Federal Savings and the Kansas Association of Teachers of Mathematics (KATM) have established a scholarship to be awarded to a practicing Kansas (K-12) teacher for the best mathematics teaching enhancement proposal. The scholarship is $1000 to be awarded at the annual KATM conference. The scholarship is competitive with the winning proposal determined by the Executive Council of KATM.

PROPOSAL GUIDELINES:

The winning proposal will be the best plan submitted involving the enhancement of mathematics teaching. Proposals may include, but are not limited to, continuing mathematics education, conference or workshop attendance, or any other improvement of mathematics teaching opportunity. The 1-2 page typed proposal should include

- A complete description of the mathematics teaching opportunity you plan to embark upon.
- An outline of how the funds will be used.

An explanation of how this opportunity will enhance your teaching of mathematics.

REQUIREMENTS:

- Have a continuing contract for the next school year in a Kansas school.
- Teach mathematics during the current year.

Be present to accept the award at the annual KATM Conference.

APPLICATION:

To be considered for this scholarship, the applicant needs to submit the following no later than **June 1 of the current year.**

- A 1-2 page proposal as described above.

Two letters of recommendation, one from an administrator and one from a teaching colleague.

PLEASE SUBMIT MATERIALS TO: Betsy Wiens, Phone: (785) 862-9433, 2201 SE 53rd Street, Topeka, Kansas, 66609

Visual Reasoning Tools in Action

The integration of appropriate tools and technology is an important guiding principle for school mathematics (NCTM 2014). Although we tend to focus on integrating technology in school mathematics, this article discusses tools that have consistently been an important part of mathematics teaching and learning.

Mathematics teachers have always encouraged their students to draw pictures or diagrams to make sense of and solve problems. Pólya (1945) includes drawing figures as a useful heuristic. The Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) identifies the strategic use of appropriate tools as one of the mathematical practices and emphasizes the use of pictures and diagrams as reasoning tools. Starting with the early elementary grades, CCSSM discusses students' solving of problems "by drawing." In later grades, such specific forms of diagrams as number lines, area models, tape diagrams, and double number lines are mentioned.

Although these diagrams may not be a common feature in U.S. mathematics curricula, they are common in many East Asian curricula. For example, Beckmann (2004) and Ng and Lee (2009) describe how strip (or bar) diagrams are used in the Singaporean elementary school curriculum. Watanabe, Takahashi, and Yoshida (2010) discuss how different visual representations, including tape diagrams and double number lines, are used in Japanese elementary mathematics textbooks. Murata (2008) articulates how a consistent use of a particular diagram helps Japanese students make sense of mathematics. One of the recommendations in *Improving Mathematical Problem Solving in Grades 4 through 8: A Practice Guide* (Woodward et al. 2012, p. 1) is to "teach students how to use visual representations." Thus, the integration of appropriate visual representations is an important step toward putting principles into action.

But how do middle-grades students use pictures and diagrams to make sense of and solve problems? This article illustrates the power of these visual reasoning tools by describing how Japanese sixth and seventh graders used a variety of pictures and diagrams to solve and make sense of problems. We will also discuss potential challenges and opportunities that these visual reasoning tools offer to middle school mathematics teaching and learning.

BACKGROUND

Recently, I had the opportunity to visit several Japanese schools, both elementary schools (grades 1–6) and lower secondary schools (grades 7–9). A typical mathematics lesson began with the teacher posing a contextualized problem without demonstrating how to solve it, as described in Stigler and Hiebert (1999). The students worked independently on the problem for about ten minutes; the majority of the class period was then spent on discussing, not just sharing, students' solutions. As I observed those mathematics lessons, I noticed students using different diagrams not only to solve the given problems but also to explain their ideas to classmates. Here are some of the ways that students used diagrams.

DOUBLE NUMBER LINE

A double number line is composed of a pair of number lines that are drawn parallel to each other and hinged at 0. Because the scaling on the two number lines is (usually) different, the model can visually show proportional relationships of two quantities. Double number lines are commonly used in Japanese elementary mathematics textbooks (Watanabe, Takahashi, and Yoshida 2010).

Table 1 Which child ran faster?

	Distance (m)	Time (sec.)
Shinya	40	8
Tohru	40	9

In one sixth-grade introductory lesson on "speed," the teacher presented table 1 and asked, "Which child ran faster?" Students easily decided that Shinya was faster. When the teacher asked them to explain how they knew that Shinya was faster, the class agreed that because both Shinya and Tohru ran the same distance (40 m), Shinya ran in a shorter time period and therefore must be faster.

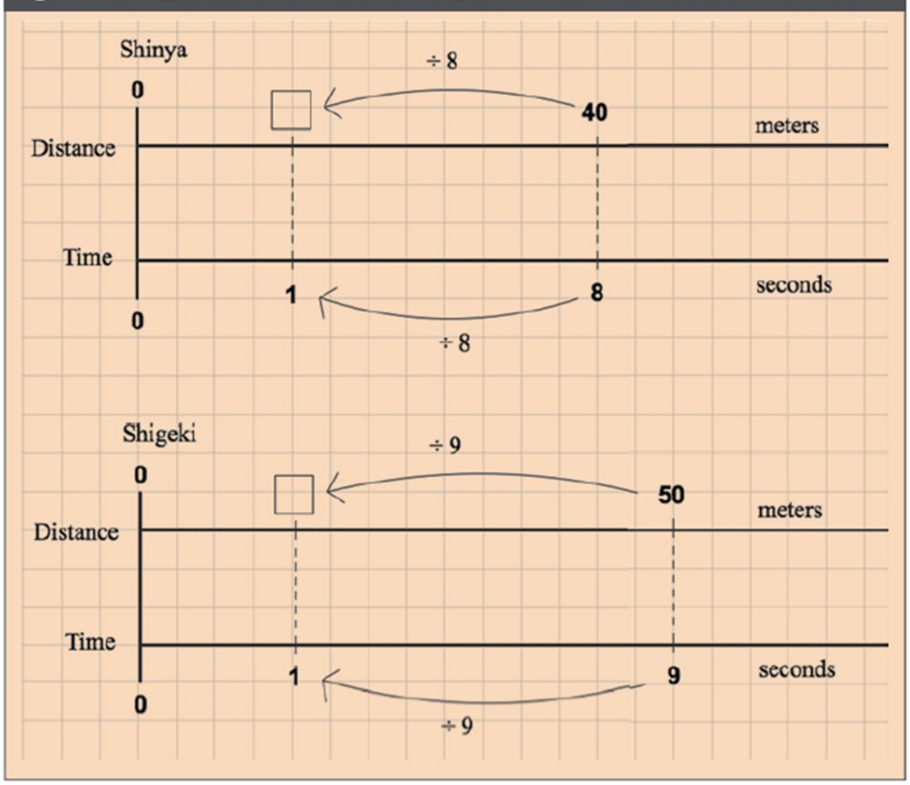
Table 2 Shigeki was determined to be the faster runner.

	Distance (m)	Time (sec.)
Shinya	40	8
Tohru	40	9
Shigeki	50	9

The teacher then presented the distance and the time for another student, Shigeki, and asked the class to determine which of these two students ran faster, Tohru or Shigeki. (See table 2.) Again, the class easily decided that Shigeki ran faster, explaining that since the two students ran for the same amount of time (9 sec.), Shigeki, who ran a longer distance, must be faster.

Then, the teacher posed the main task for the lesson, "Who ran faster, Shinya or Shigeki?" On the basis of the opening discussion of the lesson, the students realized that if they could somehow make either the time or the distance the same, they could decide

Fig. 1 A sixth-grade student used these diagrams to equate the two students' time.



which student ran faster. One student wanted to make the time the same. She decided to figure out how far each student ran in 1 second, so she drew a pair of double-number-line representations, as shown in **figure 1**. She determined that Shinya ran 40 ÷ 8 = 5 meters, whereas Shigeki ran 50 ÷ 9 = 5.6 meters (rounded to the nearest tenth). Therefore, since the amount of time was the same (i.e., 1 second), Shigeki was faster because he had run a longer distance.

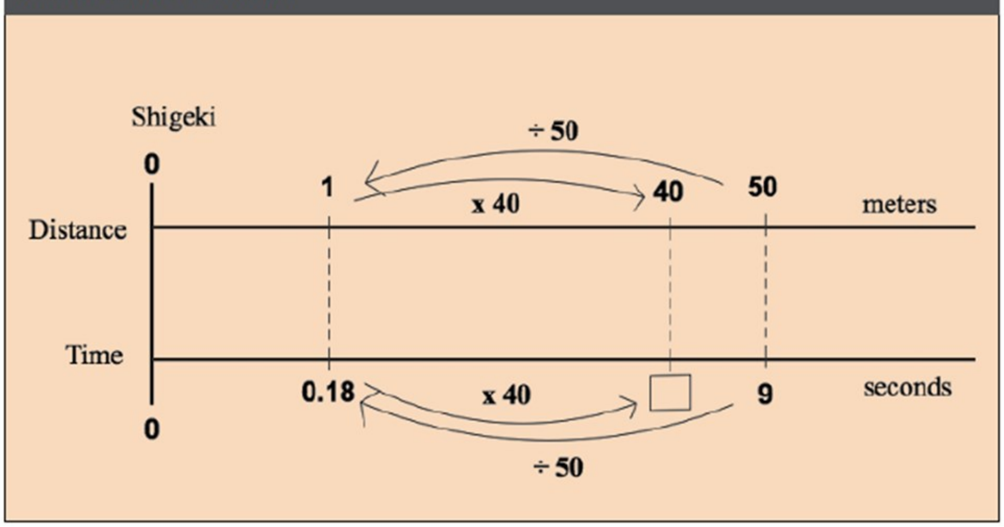
Another student wanted to make the distance the same, that is, to determine how much time Shigeki needed to run 40 meters. He drew the double number-line representation, shown in **figure 2**, and calculated Shigeki ran for 9 ÷ 50 × 40 = 7.2 sec. Since Shigeki needed less time to run 40 meters than Shinya, the student concluded that Shigeki was faster. Both of these students used double-number-line diagrams as a tool to determine what calculations were needed.

TAPE DIAGRAM

A tape diagram is another common visual representation in Japanese elementary mathematics textbooks (Watanabe, Takahashi, and Yoshida 2010) and is similar to the strip (or bar) diagrams in Singaporean textbooks. Although it can be used for a variety of situations, a tape diagram is particularly useful in situations that compare two or more quantities.

In a later lesson on speed in the same sixth-grade classroom, the teacher posted table 3. Students were able to determine that the sixth grader was a faster runner because the first grader could only run 80 meters with the time doubled to 16 seconds, whereas the sixth grader could run 100 meters in the same amount of time. The teacher then posed the following problem to the class:

Fig. 2 Another sixth grader used this diagram to determine how much time Shigeki needed to run 40 meters.

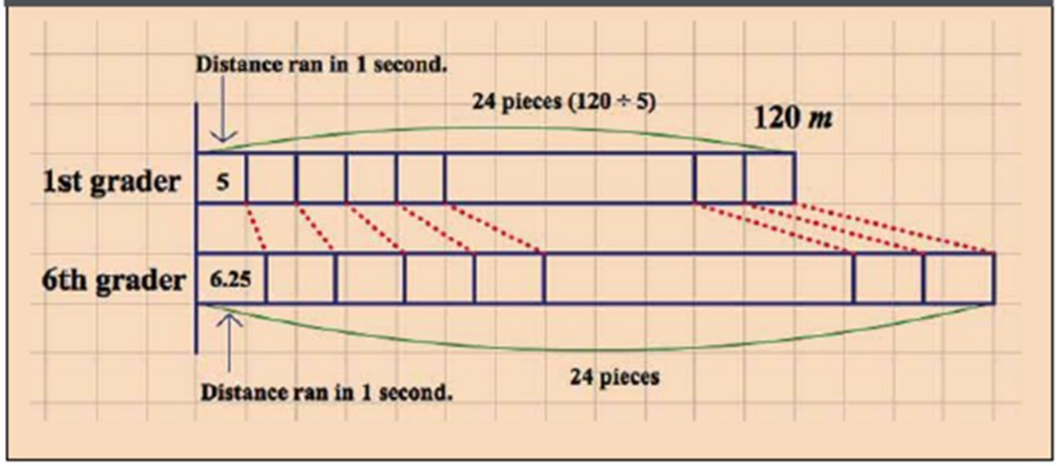


These two students will be racing on a 120 meter race course. They will start at the same time. If we want them to reach the goal at the same time, how far ahead of the sixth grader should the first grader start?

Many students used double number lines to help them figure out how far the first grader could run in the amount of time that the sixth grader could run the 120 meter course. One student misinterpreted the problem and tried to determine how many meters farther the sixth grader must run to reach the goal at the same time the first grader completed the 120 meter course. He first figured out that

that the first grader could run 5 meters in 1 second, and the sixth grader could run 6.25 meters in 1 second. From that information, he concluded that the first grader would need 24 seconds to complete the 120 meter course. He then drew the tape diagram to figure out that the sixth grader would have to run 150 meters, or 30 meters farther, than the first grader. (**Fig. 3** is the tape diagram model that the teacher drew on the blackboard, based on the student's drawing.) Had he interpreted the problem correctly, this student would have drawn a related yet different diagram. Since the sixth grader would need 19.2 seconds to run 120 meters, he would have drawn 19.2 pieces of blocks for each student. Moreover, the known distance, 120 meters, would be for the sixth grader, not the first grader, as shown in **figure 3**. Then, he would have figured out that the first grader could run 96 meters in that time span (19.2 × 5). Thus, the first grader must start 24 meters in front of the sixth grader for the two students to reach the finish line at the same time.

Fig. 3 A sixth grader used this tape diagram to determine that the sixth grader in the problem must run 30 meters farther than the first grader.



AREA MODEL

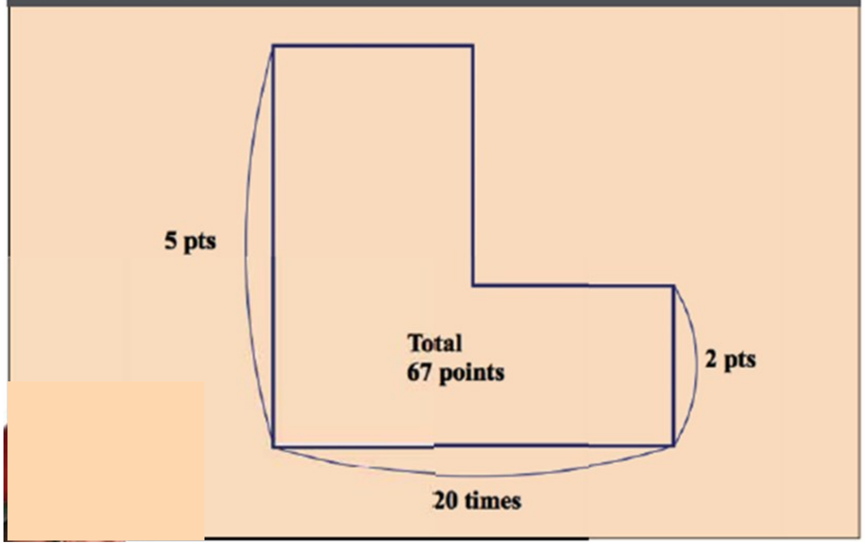
An area model is a useful model for certain multiplicative situations in which one quantity is the product of two other quantities. Readers are probably familiar with using this model to illustrate multiplication of decimal numbers or fractions. However, because it models multiplicative situations, the area model can serve as a powerful reasoning tool for solving certain types of problems involving multiplication and division. In a seventh-grade lesson on applications of linear equations, the teacher posed the following problem to his students:

> In a rock-paper-scissors game, a winner gets 5 points and a loser gets 2 points. After a man played this game 20 times, his total score was 67 points. How many times did he win, and how many times did he lose?

Because this was only the fourth lesson in the unit of linear equations, many students solved this problem without using an equation. In particular, some students used the area model shown in **figure 4** to help them solve the problem.

In this L-shaped diagram, the vertical side on the left represents the points that the player earned by winning a game; the vertical side on the right stands for the points earned by a loss. The vertical side in the middle, therefore, is the difference in points earned between a win and a loss. The horizontal dimension in **figure 4** represents the number of games played: The bottom represents the total number of games played (20); the unlabeled, top-left side corresponds to the number of games won; and the unlabeled, horizontal side to the right of the L-shape stands for the number of games lost. The "area" represents the total points that the person earned.

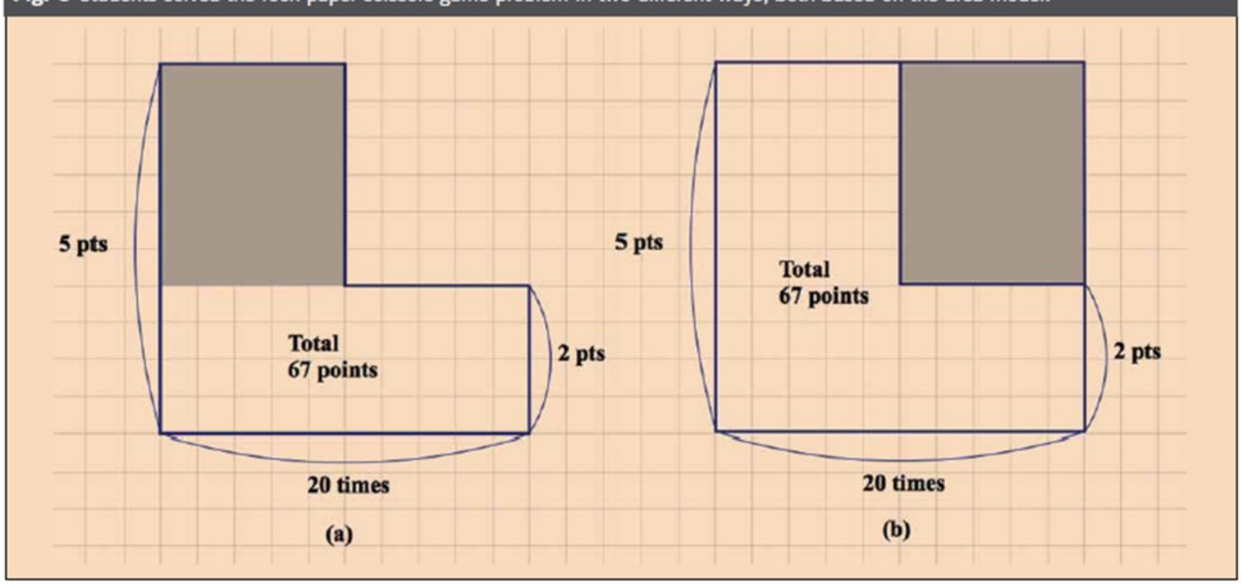
Fig. 4 Some seventh graders used this area model to help them solve the problem about the rock-paper-scissors game.



Using this diagram, some students solved the problem by noticing that the area of the shaded part in **figure 5a** is 27 (i.e., 67 – 20 × 2 = 27). Therefore, the number of games that the person won, that is, the dimension of the horizontal side on the top left, is 9 (i.e., 27 ÷ 3 = 9). Then, by subtracting 9 from the total number of games played, students found that this person lost 11 games.

Other students determined that the shaded part in **figure 5b** is 33 (i.e., 20 × 5 – 67 = 33). Therefore, the number of games that this person lost must be 11 (i.e., 33 ÷ 3 = 11). Once again, by subtraction, these students were able to figure out that 9 games were won.

Fig. 5 Students solved the rock-paper-scissors game problem in two different ways, both based on the area model.



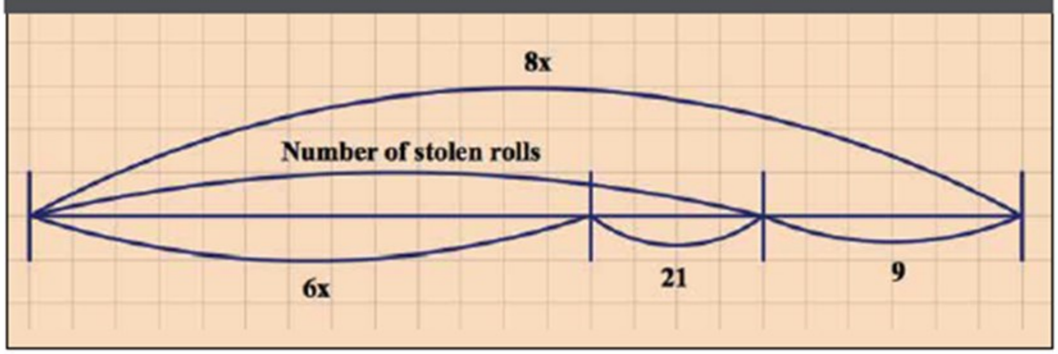
SEGMENT DIAGRAM

Segment diagrams are structurally identical to tape-strip-bar diagrams. Instead of using a thin rectangle to represent a quantity, a segment represents a quantity. Thus, any tape strip-bar diagram can be replaced with a segment diagram. A different seventh-grade teacher posed the following problem, taken from a Japanese mathematics book published in the seventeenth century:

> One night a group of thieves robbed a clothing store and stole rolls of silk. They hid under a bridge and tried to determine how they should split their loot. One thief said, "If we give 6 rolls to each, there will be 21 extra rolls left, but if we give 8 rolls to everyone, we are 9 rolls short." How many thieves were there, and how many rolls of silk did they steal?

The teacher explained where this problem originated. Although the context may be socially inappropriate in North America, it is perhaps similar to pirates sharing their loot in Western folktales or fantasies. Teachers may want to alter the problem context if they choose to use this problem in their own classrooms.

Fig. 6 Some seventh graders used a similar diagram to determine the number of thieves and the number of stolen rolls.



Similar to the other seventh-grade class, these students had just recently been introduced to linear equations. Thus, some students used a diagram like the one shown in **figure 6** as they thought about the problem. These students let x be the number of thieves. From this diagram, those students realized that the difference between $8x$ and $6x$ is the sum of 21 and 9. Then, they solved a simple linear equation, $2x = 30$, to find the number of thieves, 15. After that, they calculated the number of stolen rolls to be 111 (i.e., $6 \times 15 + 21 = 111$, or $8 \times 15 - 9 = 111$).

CHALLENGES AND OPPORTUNITIES

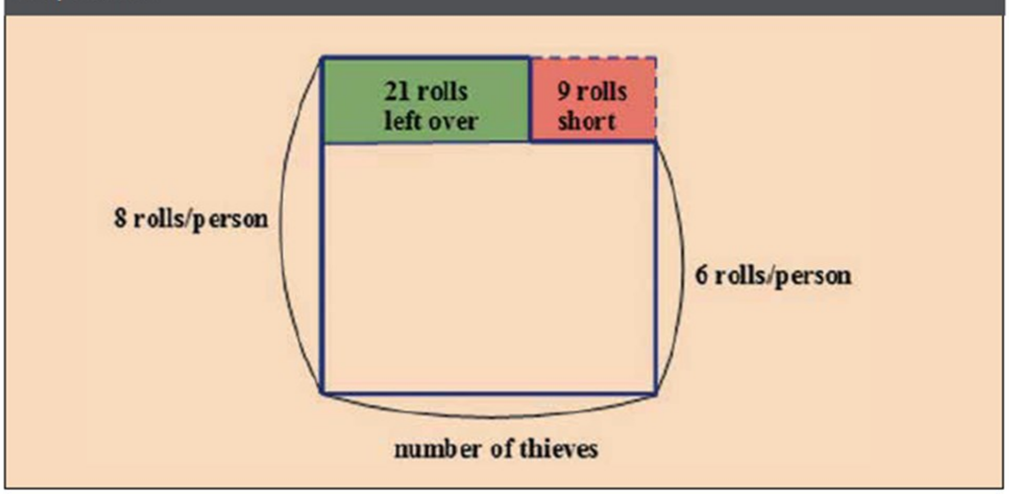
The examples discussed illustrate the power of drawings and diagrams as problem-solving strategies and as explanation tools. However, one of the reasons that the students discussed in these examples were able to confidently use these visual representations as tools is because they had been using them since their early elementary school years (Murata 2008; Watanabe, Takahashi, and Yoshida 2010). Thus, one of the challenges for middle-grades mathematics teachers in North America, where these tools are not used as consistently, is how to encourage students to include visual representations in their own reasoning toolkits. One suggestion is to incorporate visual representation tools intentionally when reviewing materials from prior grades. Teachers could give students a hypothetical student's solution to a problem that uses a visual representation tool and ask them to explain how the visual model represents the problem situation. Then, they could follow up by using visual representations to solve other problems.

Because these visual representations are powerful problem-solving tools, they also present a different challenge to mathematics teachers. As discussed, the two seventh-grade classrooms in which students used visual representations were both focusing on applications of linear equations. The teachers hoped that the contextualized problems they posed to the class would help students see the usefulness of linear equations as a problem-solving tool. However, some students already possessed powerful tools and did not need to use linear equations to solve the problems.

Similarly, double-number-line diagrams can be used to represent missing-value proportion problems. Thus, instead of setting up proportions to find the missing value, students can solve the problems arithmetically as prospective elementary school teachers did in Watanabe, Takahashi, and Yoshida (2010). Therefore, the challenge is how to help students who are proficient with these visual reasoning tools learn the power and usefulness of more advanced mathematical tools like linear equations.

As mathematics teachers consider this challenge, keep in mind that these visual representations are adequate when the focus is on solving particular problems. In other words, if the classroom discussion focuses only on the correctness of the answers, these solution strategies would be equally valid. The students would have no motivation to understand or appreciate other strategies, such as linear equations, even if those strategies are more advanced mathematically. On the other hand, class discussions that extend beyond the correctness of answers or build on student solutions—for example, comparing and contrasting a variety of solution strategies or examining mathematical structures of problem situations—may be enriched by these visual representation tools. They may help students make sense of more advanced mathematics. Making connections among various mathematical representations and facilitating meaningful mathematical discourse are two key features of effective teaching of mathematics (NCTM 2014).

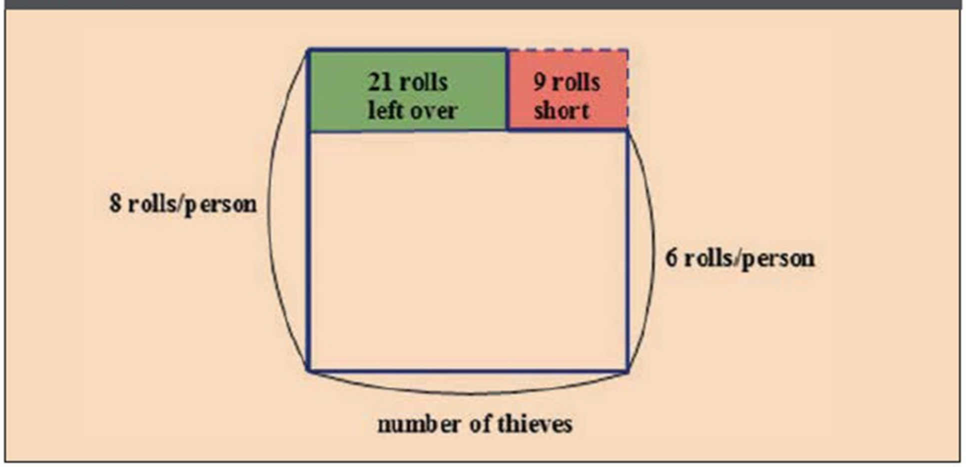
Fig. 7 The teacher anticipated that some students might use the area model to solve the problem.



These visual representations also offer new opportunities for exploring mathematical relationships. Although none of the students in the second seventh-grade classroom actually used the area model to solve the Silk Thieves problem, the teacher anticipated that some students might use the area model shown in **figure 7**. In this diagram, the vertical dimension represents the number of rolls for each thief, whereas the horizontal dimension represents the number of thieves. The shaded rectangle on the top left represents the extra rolls left over when 6 rolls were given out to all the thieves, and the shaded rectangle on the right represents the 9 roll shortage if 8 rolls were to be given out to the thieves.

Figure 8 shows the two sharing situations separately. The area model shown in **figure 7** can be thought of as the combination of the two models in **figure 8**. From this diagram, the sum of 21 and 9 must equal the product of 2 (the difference in the number of rolls to be given to each thief) and the number of thieves. Therefore, the number of thieves must be 15. Once the total number of thieves is known, we can calculate the total number of stolen rolls to be 111, either by 15 × 6 + 21 or 15 × 8 – 9.

Fig. 7 The teacher anticipated that some students might use the area model to solve the problem.



The fact that this problem can also be represented using the area diagram suggests that the rock-paper-scissors game problem and the Silk Thieves problem have a common mathematical structure. Therefore, teachers could have students explore that mathematical structure. Then, students could be challenged to represent the rock-paper-scissors game problem using the segment diagram. Exploring and using mathematical structures is an important mathematical practice highlighted by CCSSM.

How are these problems related? In the rock-paper-scissors game problem, a common reasoning strategy for solving the problem would be to think about the situation if the man won all his games. In that case, that individual would have earned 100 points, 33 more points than he actually earned. (See the shaded rectangle in **fig. 5b**.) These are the points that "fell short." However, had he lost all the games, this person would have earned 40 points, when actually 27 more points were earned. These are the "leftover" points. Thus, a parallel to the Silk Thieves problem in the context of rock-paper-scissors would be this:

> A student played the game several times. His total score was 27 more points than what he would have earned had he lost all the games, and it was 33 fewer points than what he would have earned had he won all the games. How many games did the student play, and what was the score?

VISUAL REPRESENTATIONS AS POWERFUL TOOLS

We considered how middle-grades students can use visual representations as powerful problem-solving tools. These examples reaffirm the importance of the systematic and consistent integration of visual representations in mathematics teaching suggested by other scholars. Thus, integrating appropriate visual reasoning tools is one way to make the mathematics teaching principles (NCTM 2014) come to life. Of course, this integration depends on the teachers' ability to use these tools.

Getting Started

The following resources are for readers who are interested in learning more about the specific visual representations explored in this article:

- Beckmann (2004) and Ng and Lee (2009) discuss how strip/bar diagrams are being used in Singaporean textbooks.
- Cohen (2013) describes how strip diagrams can be used to help students make sense of proportions. Keep in mind that any tape/strip/bar diagram can be replaced with a segment diagram.
- Beckmann and Fuson (2008) explore how various mathematical representations, including tape diagrams and double number lines, can be useful in middle-grades mathematics classrooms.
- Watanabe, Takahashi, and Yoshida's (2010) offering includes a detailed discussion on double number lines.

Although area models have been used in U.S. mathematics curricula, very few resources discuss how to use them as reasoning tools. A key feature of area diagrams is that they represent three quantities: One of the quantities is the product of the other two. Interested readers may want to keep this feature in mind and try to identify word problems that involve quantities related in this manner.

Note that an English translation of grades 1 through 9 Japanese mathematics textbooks, *Mathematics International* (Tokyo Shoseki 2012), is available in North America.

If you are not familiar with any of these visual reasoning tools, I encourage you to try them because solving problems will acquaint you with their mechanics. The **sidebar** provides some resources for getting started. A deep examination of these tools will prepare you for helping students see both their power and limitations (NCTM 2014). It will also help students develop the mathematical practice of using appropriate tools strategically (CCSSI 2010).

I hope this article will motivate many teachers to collaborate and devise plans to integrate these tools in such ways that they build student understanding and reasoning (NCTM 2014).

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Reconciling Representations

Sometimes a student's unexpected solution turns a routine classroom task into a real problem, one that the teacher cannot resolve right away. Although not knowing the answer can be uncomfortable for a teacher, these moments of uncertainty are also an opportunity to model authentic problem solving. This article describes such a moment in my class Problem Solving for Teachers. It started when I (coauthor Zahner) introduced this calendar problem from *Mathematics Teacher*:

> Triangles with sides (a, b, c) are randomly generated in the following manner: $c = 1$, $0 < a \le 1$, and $0 < b \le 1$. Any value of (a, b, c) that does not satisfy the triangle inequality theorem, $a + b > c$, is discarded. What is the probability (to the nearest hundredth) that a random triangle is obtuse? (McLoughlin 2002 p. 30)

I found this problem fun, and constructing the solution led me to make surprising connections among algebra, geometry, and probability. Through class problem solving and discussion of solutions (following principles from Smith and Stein 2011), I planned to connect the geometry and probability standards in the Common Core State Standards for Mathematics (CCSSI 2010) with the NCTM Standards and the Standards for Mathematical Practice (SMPs).

My plans changed when two students created different representations that appeared to result in contradictory solutions. Reconciling these solutions led my students and me on a multi-week quest to understand what was going on. Ultimately, this process of inquiry was a powerful lesson about the problem-solving practices as envisioned in the NCTM's Standards (2000) and in the Common Core State Standards for Mathematics (CCSSI 2010).

Coauthor Dent, then a mathematics teacher at Boston University Academy and a student in Calculus for Teachers, reconciled the solutions using multivariable calculus, with help from one of his high school students. Here we share the story of this problem, reflect on lessons that we learned about teaching through problem solving, and discuss pedagogical strategies that helped us navigate uncertainty in our mathematics classrooms.

AN EXPECTED SOLUTION

In planning to teach this problem, I solved it by choosing trial values for a and b and looking for patterns (indicating a nice connection with the SMP 8: "Look for and express regularity in repeated reasoning"). For example, if $a = 0.7$ and $b = 0.5$, then the triangle has side lengths 0.7, 0.5, and 1. Using the converse of the Pythagorean theorem, this triangle is obtuse because 0.72 + 0.52 = 0.74 < 12.

To generalize, I needed to represent all possible solutions. One possibility would be to graph points with coordinates (a, b) in the unit square. Right triangles correspond to points on the quarter circle a2 + b2 = 1 in the first quadrant. The restriction $a + b > 1$ excludes points in the triangle with vertices (0, 0), (1, 0), and (0, 1). Finally, the converse of the Pythagorean theorem suggests that, if *a*2 + *b*2 < 1, then the triangle is obtuse; and if *a*2 + *b*2 > 1, then the triangle is acute. Combining this information yields the graph shown in **figure 1**.

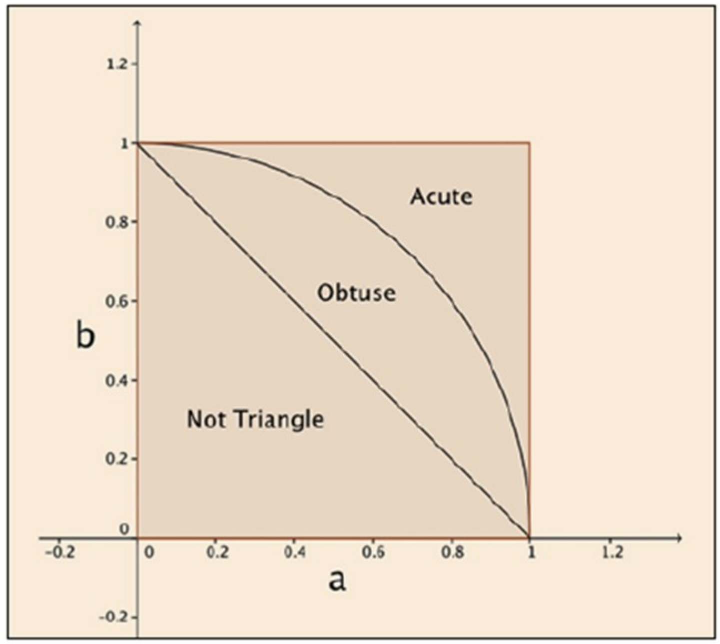


Fig. 1 A graphical representation shows possible pairings of *a* and *b* with regions indicating the type of triangle formed.

The region of non-triangles has an area of 1/2. Excluding non-triangles, the probabilities of forming obtuse and acute triangles can be found by dividing the area of each region by 1/2. Interestingly, the probability of forming a right triangle is zero, even though infinitely many right triangles are possible. One way to understand this paradoxical result is to imagine fixing *a* and asking, given *a*, what is the probability that a randomly chosen value for *b* will result in a right triangle? For any given value of *a* (e.g., 1/2), there is exactly one value of *b* that will form a right triangle (in this case, 13/2). However, the probability that *b*, a continuous random variable, takes on a particular value in its domain is zero. Since this is true for all *a* in $0 < a \leq 1$, the probability that random choices of a and b will result in a right triangle is zero.

The area of each region in **figure 1** follows:

Area(*Not Triangle*) = 1/2 = 0.50

Area(*Obtuse*) = p/4 – 1/2 ≈ 0.29

Area(*Acute*) = 1 – p/4 ≈ 0.21

Area(*Right*) = 0

Therefore, to answer the problem, the probability of generating an obtuse triangle is

$$\frac{(\pi / 4 - 1/2)}{1/2} \approx 0.57.$$

ALTERNATIVE SOLUTIONS

After I introduced this problem, my students solved it in groups. Most groups started by trying a few values of *a* and *b* to explore relationships, and two groups created a table showing *a*2 + *b*2 for different values of *a* and *b*. **Figure 2** shows one sample table in which the cells are color-coded by the type of triangle generated for each combination of *a* and *b* (red for non-triangles, blue for obtuse triangles, green for acute triangles, and yellow for right triangles). I encouraged these groups to consider what would happen on the borders where the color changed (e.g., when a = 0.357 and b = 0.907)to shift from the discrete case to the continuous case. The color-coded table of **figure 2** suggests the graph in **figure 1** (after a flip). Therefore, during the whole-class discussion of this problem, I planned to ask one student to share the table to discuss how organizing repeated calculations can help solve this problem.

a\b	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.02	0.05	0.1	0.17	0.26	0.37	0.5	0.65	0.82	1.01
0.2	0.05	0.08	0.13	0.2	0.29	0.4	0.53	0.68	0.85	1.04
0.3	0.1	0.13	0.18	0.25	0.34	0.45	0.58	0.73	0.9	1.09
0.4	0.17	0.2	0.25	0.32	0.41	0.52	0.65	0.8	0.97	1.16
0.5	0.26	0.29	0.34	0.41	0.5	0.61	0.74	0.89	1.06	1.25
0.6	0.37	0.4	0.45	0.52	0.61	0.72	0.85	1	1.17	1.36
0.7	0.5	0.53	0.58	0.65	0.74	0.85	0.98	1.13	1.3	1.49
0.8	0.65	0.68	0.73	0.8	0.89	1	1.13	1.28	1.45	1.64
0.9	0.82	0.85	0.9	0.97	1.06	1.17	1.3	1.45	1.62	1.81
1	1.01	1.04	1.09	1.16	1.25	1.36	1.49	1.64	1.81	2

Fig. 2 Calculated values of $a^2 + b^2$ are used to determine the type of triangle formed.

As I anticipated, several groups made a diagram similar to that shown in **figure 1**. One student who made this solution was Katie, and I planned to ask her to share her solution next.

In addition, one group constructed a representation that I had not anticipated. Claudine recalled that any triangle inscribed in a semicircle is a right triangle. Using this fact, she drew a one-unit segment (for the side with length c = 1) and constructed a semicircle using c as a diameter. Finally, she constructed a triangle with side lengths a and b built on opposite ends of the segment with length c; she called the intersection of these two segments point E. Drawing a few sample triangles, Claudine reasoned that any triangle in which E landed on the semicircle would be a right triangle (see **fig. 3a**), any triangle with E inside the semicircle would be obtuse (see **fig. 3b**), and any triangle in which E was above the semicircle would be acute (see **fig. 3c**). Non-triangles did not appear in Claudine's representation.

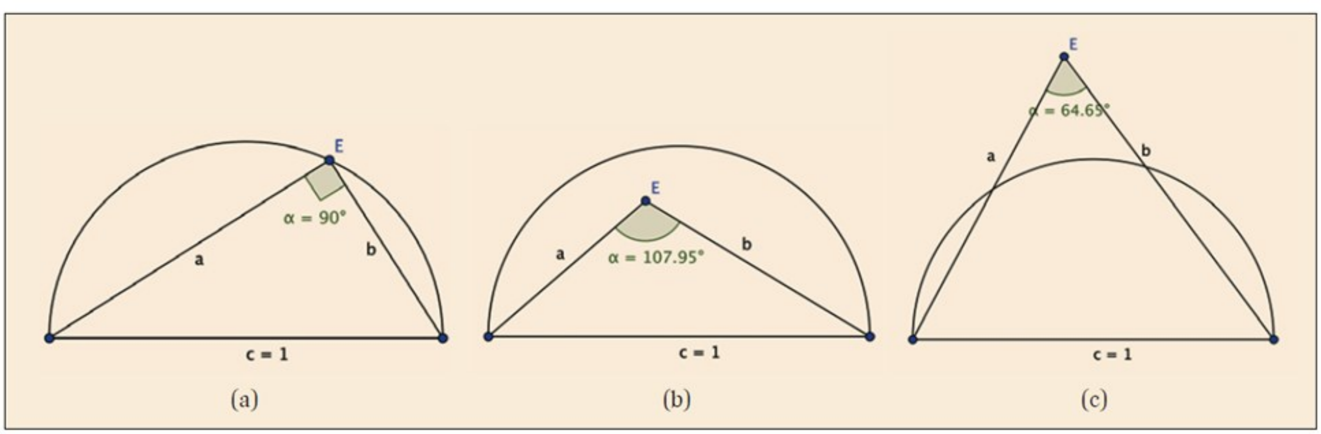


Fig. 3 Claudine's representation shows three distinct possibilities: a right triangle (**a**); an obtuse triangle (**b**); and an acute triangle (**c**).

Claudine's group mates helped find bounds for the possible location of *E*. The maximum lengths of *a* and *b* are 1, so the students constructed arcs of length 1 on the ends of segment *c* to set bounds for the vertex point *E* (essentially following the process for constructing an equilateral triangle). The result is an arch with an inscribed semicircle (see **fig. 4**).

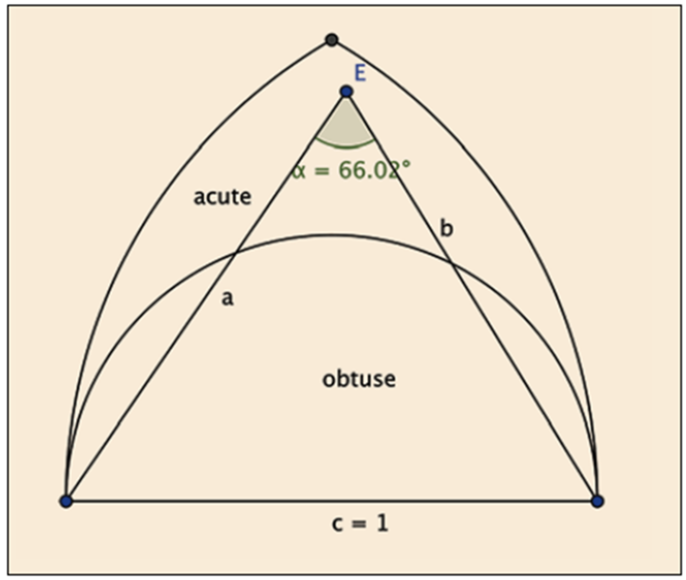


Fig. 4 Regions for the location of point *E* are associated with the type of triangle formed.

Once I understood Claudine's diagram, I appreciated how it connected with the geometry of the random triangles. I asked her group to find the probabilities, and I walked away to help another group. To find the probabilities, Claudine and her group mates also calculated the area of each region, a more challenging task than finding the areas in figure 1. The area of the semicircle in which obtuse triangles are formed is

$$Area = \frac{\pi r^2}{2} = \frac{\pi (1/2)^2}{2} \approx 0.39.$$

The first step for finding the area of the region in which point *E* lands for acute triangles is to find the area of the arch. The arch is made of an equilateral triangle (region 1 in **fig. 5**) and two segments of a circle (regions 2 and 3 in **fig. 5**). The equilateral triangle's area is 13/4. The area of each segment of a circle can be found by computing 1/6 of the area of the circle and subtracting the area of the triangle. Putting these pieces together, Claudine's group found that

$$Area\ of\ arch = \left(\frac{1}{6}\pi - \frac{\sqrt{3}}{4}\right) + \left(\frac{1}{6}\pi - \frac{\sqrt{3}}{4}\right) + \frac{\sqrt{3}}{4}$$
$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} \approx 0.61.$$

Finally, they found the area of the "acute region" in **figure 4** by subtracting the area of the semicircle: 0.61 – 0.39 = 0.22.

THE UH-OH MOMENT

Gathering the class together, I asked Tyler, Katie, and Claudine to share their solutions. I was delighted that Claudine proposed to share her solution because it gave us three alternative representations to compare: a numerical table, a graph, and a geometric representation. I planned to sequence the discussion to highlight these differences.

First, Tyler shared his group's table. He discussed the fact that the table representation helped his group make sense of the problem but did not necessarily yield exact probabilities.

Next, Katie showed her solution (see **fig. 1**). She used decimal values for the probabilities:

$P(\text{No Triangle}) = 0.5$

$P(\text{Obtuse}) = 0.29$

$P(\text{Acute}) = 0.21$

$P(\text{Right}) = 0$

Katie's direct use of the areas in **figure 1** as probabilities led to a short discussion about how the "correct" answer to this question depended on interpretation of the question—that is, does the probability need to account for non-triangles?

Finally, Claudine presented her diagram and shared her answers:

Area of obtuse region = 0.39

Area of acute region = 0.22

At this point, things got interesting. I had not worked out Claudine's solution in advance, so I had not anticipated that Katie and Claudine would have different answers. After double-checking our calculations, one student suggested that there was some kind of stretching going on and that we had to compare the ratio of the areas in each solution. Even though the numbers differed, this student argued, the ratio would be constant. This approach sounded reasonable, but we found the following results:

0.285 : 0.215 = 1.326 : 1 (Katie's solution) and

0.393 : 0.221 = 1.778 : 1 (Claudine's solution).

Further verifying that the ratios could not be the same, one student pointed out that Katie's areas were in terms of multiples of p alone, whereas Claudine's areas included multiples of p and 13. A simple scaling could never reconcile these solutions.

At this point, I was not sure what to do. This Random Triangle problem, which I thought I understood, had become a real problem. We put this problem in the "parking lot" of issues that we would return to later.

OBSERVATION AND COMPARISON

Following Erickson's (2001) method of using random variables to investigate perimeter-area relationships in rectangles, I created a dynamic simulation in GeoGebra to explore this problem. To construct Katie's solution, I generated random decimals between 0 and 1 for a and b and plotted A = (a, b). The color of A was determined by a rule checking whether the triangle with side lengths a, b, and 1 was acute (green), obtuse (blue), or impossible (red).

To create Claudine's solution, I followed her geometric construction, starting with a segment from (0, 0) to (1, 0) in the second graphics window of the same GeoGebra file. Using the same values of *a* and *b* from the Katie's solution, I constructed a circle with radius *a* centered on the point (0, 0) and a circle with radius *b* centered on (1, 0). Point *E* was constructed to be the intersection of the two circles in the first quadrant and was connected to the points (0, 0) and (1, 0). The color of *E* was green if the vertex angle at *E* was acute or blue if it was obtuse. The simulation is available online at **www.nctm.org/mt066**.

Turning on the trace feature for point *A* in Katie's solution and point *E* in Claudine's solution created the expected pattern. Because the sketch was a simulation, GeoGebra could sample a few—or many—values for *a* and *b*, and the images from **figures 1** and **4** would emerge. **Figure 6** shows the simulations after 202 and 10,005 trials.

After 10,000 trials, I noted the lack of points along the segment from (0, 0) to (1, 0) in Claudine's solution. There was no corresponding gap between the random points in Katie's solution. I let the simulation run several times to convince myself that this gap in points was related to the model rather than to random variation. Something was going on, although at this point I was not sure whether I was seeing a rounding error or something deeper.

AN AHA! MOMENT

The day after I created the simulation, my Calculus for Teachers class was discussing applications of integrals. As my students worked on a set of problems, I mentioned the surprising conundrum that we had found in my Problem Solving course. I wondered aloud whether we could reconcile the solutions using integration because there had to be some kind of density function underlying Claudine's solution. This insight was inspired by an exercise in our textbook that required setting up and integrating a population density function to find the population of a circular city (Finney et al. 2006, p. 387, exercise 23). Of course, mentioning an unsolved problem to a group of mathematics teachers is sure to spark interest. They wanted to hear more, so I showed the students the problem and my GeoGebra simulation.

Coauthor Dent, a student in my calculus class, was so interested in the conundrum that he presented this problem and the simulation to his high school students at Boston University Academy. One of Nick's students, Oliver, created a graph to illustrate what was going on in Claudine's solution. He first set up and solved simultaneous equations for the intersection of two circles:

$$y = \sqrt{a^2 - x^2} \qquad (1)$$

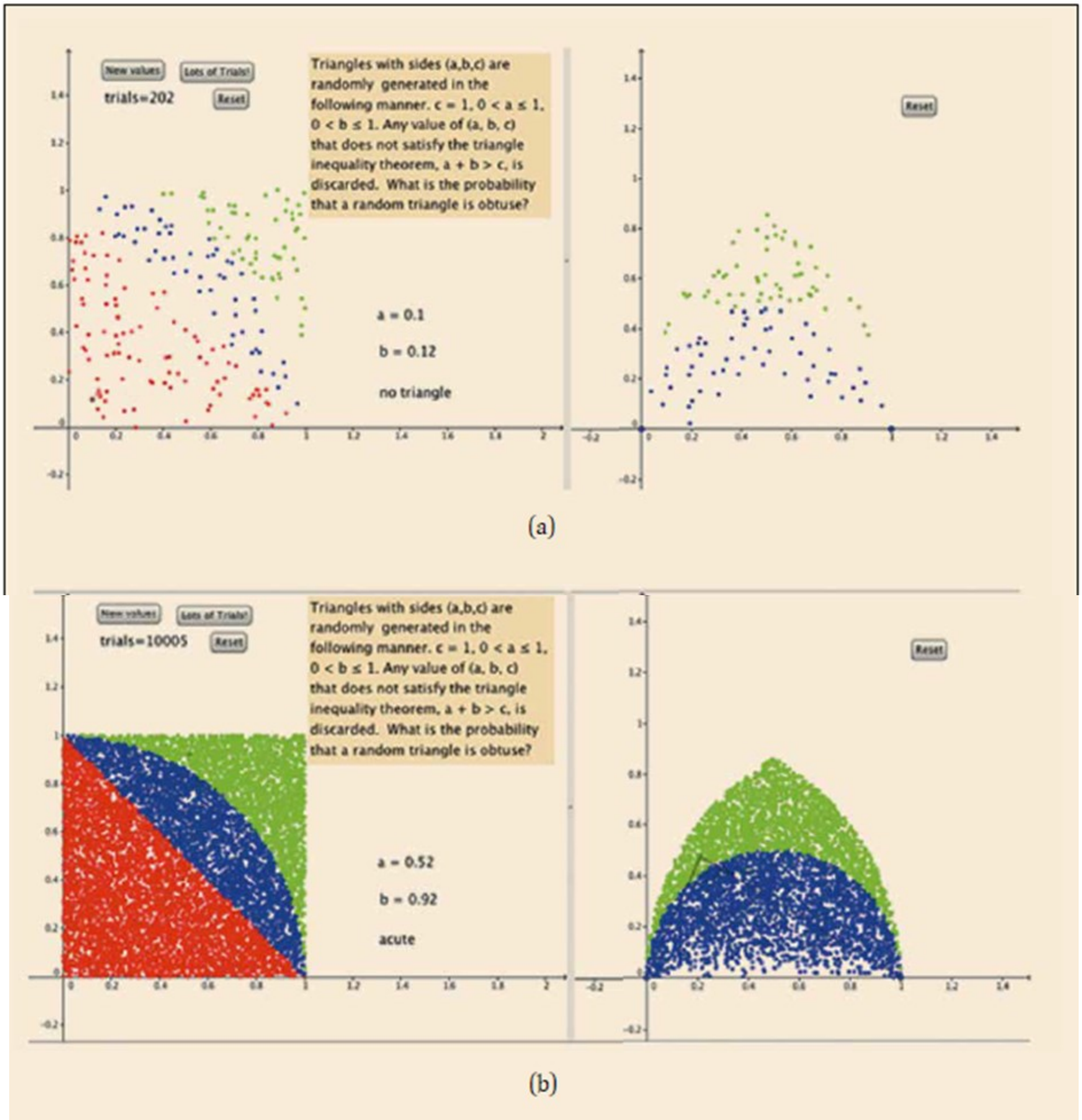
$$y = \sqrt{b^2 - (x-1)^2} \qquad (2)$$

Setting (1) and (2) equal and solving for x yields

$$x = (1/2)(1 + a^2 - b^2).$$

Substituting x back into (1) gives y in terms of a and b:

$$y = \sqrt{a^2 - \left(\frac{1}{2}\left(1 + a^2 - b^2\right)\right)^2}$$



ig. 6 The GeoGebra simulation tracks vertices *A* and *E* for 202 trials (**a**) and 10,005 trials (**b**).

Using these values for (x, y), Oliver generated a list of points in GeoGebra by letting a and b vary from 0 to 1 in increments of 0.05. Oliver's graph is reproduced in **figure 7b** with Claudine's image superimposed. For comparison, the same set of points (a, b) are also graphed on Katie's solution. The key feature revealed in **figure 7a** is that the distribution, or *density*, of points in Claudine's solution is not uniform: Point E is more likely to land in some regions than others in Claudine's diagram. This was not the case in Katie's diagram, in which all the points are uniformly distributed. The uneven distribution of points explained why Claudine's correct area calculations did not yield the correct probabilities. I brought this image to class, and we discussed how geometric probability problems rest on the assumption (often unstated) that points are evenly distributed.

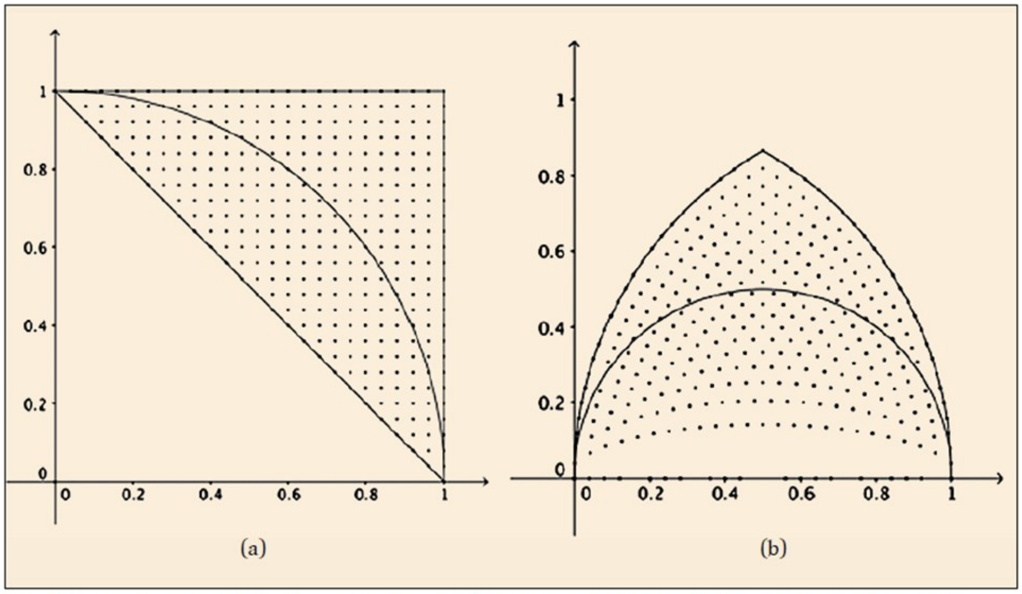


Fig. 7 The graphic generated by Oliver shows the consistent spacing of points *A* in Katie's solution and the variable spacing of points *E* in Claudine's solution.

PERSEVERANCE AND PROOF

As we revisited this problem in both my courses, Oliver's images helped us see how to reconcile the solutions. The spacing of points just above the *x*-axis in Claudine's solution shows that the gap that we observed in the simulation was not due to a rounding error. Instead, a very small range of values for *a* and *b* values result in triangles whose vertex, *E*, is located just above the segment for side *c*. Further, the pattern of points in Oliver's representation pointed the way to setting up a density function because it was reminiscent of Handa and Yakes's (2010) analysis of the distribution of random points on a dartboard.

Dent reasoned that if we could find a function to define the density of points in Claudine's solution, then we could integrate the density function to recover the probabilities. As with the traditional Dartboard problem, the density of points is a function of two variables, so this situation was more complicated than the one-variable probability densities that we were discussing in my calculus course. Nick happened to be enrolled in multivariable calculus, and he dove in to solve the problem (Nick's solution appears in the **appendix** at the end of this article).

While Dent wrestled with defining and integrating the density function, the mystery of the Random Triangle problem inspired several of my students to share this problem with friends and in their high school mathematics courses. For example, Danielle, who was student teaching in a high school geometry class, presented this problem along with the GeoGebra simulation to her students. Danielle's cooperating teacher doubted whether high school geometry students could tackle such a difficult problem, but she allowed Danielle to try it out. Much to the teacher's surprise, the students were able to arrive at the solution by following a process similar to what we did in my Problem Solving course. With some guidance from Danielle, they started by guessing a few values and then generalized by creating a graph like that shown in **figure 1**. Danielle's students felt very accomplished when she shared that *she* was working on this problem in her college mathematics class.

REAL PROBLEM SOLVING

This article has told the story of how one student's unique solution to a problem created new opportunities to explore connections among geometry, algebra, probability, and calculus, both single and multivariable. The dynamic simulation provided an inspiration and motivation for persevering with this problem and making sense of what was happening.

As a teacher, I noticed that once we had a real mystery to ponder, my students' interest in this problem grew dramatically. Students like Danielle shared this problem with their high school students and used the fact that she was still working on understanding Claudine's solution to spark discussion. Claudine and Nick both took this problem to their university-level mathematics classes. Ultimately, this relatively simple problem proved to be a rich source of mathematical exploration and growth for all of us, across the mathematical spectrum.

One of my goals for presenting this problem was to show my students the Common Core's SMP 1—"Make sense of problems and persevere in solving them"—in action. This problem succeeded beyond my expectations because Claudine's unexpected solution repositioned me: I no longer knew all the answers, and my students could see me as a problem solver. In fact, Claudine's solution pushed me to create the GeoGebra simulation and to explore this problem in far more depth.

Not knowing the answer can be an uncomfortable space for a teacher. However, a real problem is also an invaluable opportunity to model for students what problem solving—real problem solving—looks like. According to feedback from my students, the struggle and uncertainty paid off. Seeing me struggle with a problem alongside my students showed them that problem solving is possible—and powerful.

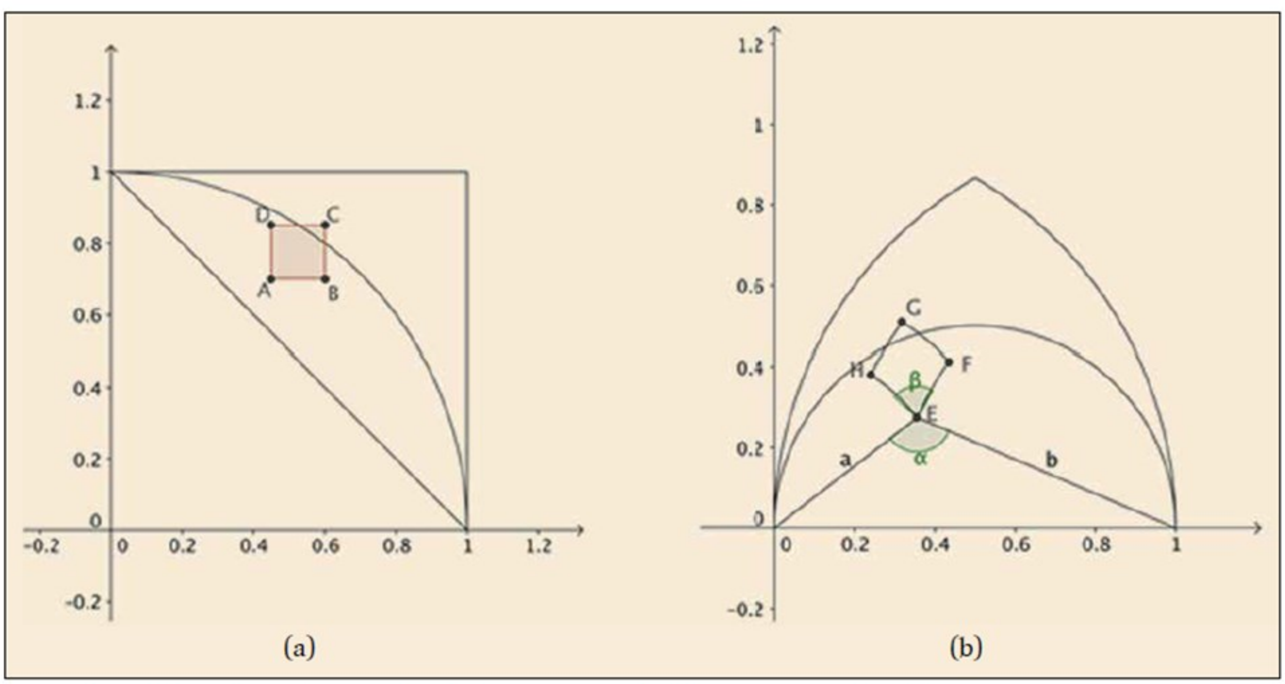


Fig. 8 The set of points inside square *ABCD* (**a**) corresponds to the region inside *EFGH* (**b**).

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NCTM Update

My name is Stacey Bell and I am pleased to be the NCTM Rep for KATM. NCTM has a new website design and has been focusing on developing its Affiliate Site for its members. As an affiliate of NCTM, KATM is able to now post our upcoming events on this new site for neighboring states to see. And likewise, we are able to see what other affiliates are doing around us. You should check it out at http://www.nctm.org/affiliates/

In other news, the NCTM election results are in.

Board of Director for the Elementary Level - Gina Kilday, Metcalf Elementary School, Exeter, RI

Board of Director for the Middle School Level - Kevin J. Dykema, Mattawan Middle School, Mattawan, MI

Board of Director for the Canadian Region - Olive Chapman, University of Calgary, AB

Board of Director for At-Large (one will be elected) - Kay A. Wohlhuter, University of Minnesota Duluth, Duluth, MN

Being members of both NCTM and KATM allows you to have a wealth of resources for you and your classroom. For this issue, I would like to highlight some video resources found on NCTM's website about why conceptual teaching is so important and other videos related to the Common Core. Below is a description from NCTM's website. (http://www.nctm.org/Standards-and-Positions/Common-Core-State- Standards/Teaching-and-Learning-Mathematics-with-the-Common-Core/)

NCTM and The Hunt Institute have produced a series of videos to enhance understanding of the mathematics that students need to succeed in college, life, and careers. Beginning in the primary grades, the videos address the importance of developing a solid foundation for algebra, as well as laying the groundwork for calculus and other postsecondary mathematics coursework. The series also covers the Standards for Mathematical Practice elaborated in the Common Core State Standards for Mathematics and examines why developing conceptual understanding requires a different approach to teaching and learning.

- Building Conceptual Understanding for Mathematics
- Mathematics in the Early Grades
- Developing Mathematical Skills in Upper Elementary Grades
- Mathematical Foundations for Success in Algebra
- Preparation for Higher Level Mathematics
- Standards for Mathematical Practice
- Parents Supporting Mathematics Learning
- Conversations about K-12 Mathematics Education (Five-Part Series)

These videos will be helpful for all stakeholders, including parents. I encourage you to check them out.

KLFA: August Press Release

KLFA Hears KSBE Listening Tour Results/Establishes Focus for 2015-16

Christy Levings, former KLFA Chair, facilitated the meeting as over 30 members developed project plans for the 2015-16 year. By consensus the group established three areas of work:

- Student Success
- Community Engagement
- Professional Learning

Members joined one of the three groups and further focused their work on three key questions: What are the first steps we must complete, what are the next steps, and what needs to be accomplished by June 2016?

Kathy Busch, State Board KLFA liaison, and Dr. Randy Watson, Commissioner of Education, presented a partial summary of the KSDE Listening Tour and vision for Kansas Schools.

The feedback from over 1000 participants concluded succeeding in life takes more than traditional academic skills. When asked, "What are the skills, attributes and characteristics of a successful 24 year old Kansan?" participants identified qualities such as:

- Academic Skills (23%)
- Non-Academic Skills (70%)
    - Conscientiousness – the tendency to be organized, responsible, and hardworking
    - Good citizenship
    - Agreeableness – the ability to get along and work well with others
- Health, Mental and Physical (3%)
- Employed (2%)
- Credentials (2%)

Kansas Learning First Alliance (KLFA) met October 13 at the Kansas Association of School Board (KASB) building.

Members approved Kansas Research and Education Network (KanREN) as a KLFA member. KanREN works to connect technology needs of its members to better serve the needs of Kansas educational institutions. Welcome KanREN!

Rebecca Lewis of Circles presented Generational Poverty: Building Kids, Strengthening Families, Changing Communities to KLFA. Rebecca shared her personal story and her calling to work with schools and communities to empower our poorest kids and parents. Some key points from her presentation included:

- Allies – People who won't let you quit.
- Community – What does this word mean to the poor? Poverty has a community but missing sense of pur pose, resources, less-supportive, and no link to opportunity.
- Working – I work but I'm still in poverty. "I don't want a handout; I want to know how to get out."
- Toxic Stress - #1 villain of poverty. www.developingchild.harvard.edu
- Development of Adults – Focus on the development of adults of the young children to make lasting changes.
- 2 by 10 – Give 2 minutes for 10 days and you will build lasting relationships with students.

More information on Circles is found at www.circlesusa.org and through Rebecca's Facebook page (Rebecca Lewis Poverty Edge) or Twitter (@povertyedge).

Marcus Baltzell, KNEA Communications Director, presented "Storytelling: The Value of Telling Our Story to Move Our Message". We tell stories to inform, inspire, validate, demonstrate, and connect. Ask the community to join us to support Kansas Public Education (www.joinus.org).

Members worked in their focus areas to weave stories into their message. Members also set a timeline of specific next steps. The three KLFA Focus Areas are:

- Student Success
- Professional Learning
- Community Engagement

CALL FOR SUBMISSIONS

Your chance to publish and share your best ideas!

The KATM Bulletin needs submissions from K-12 teachers highlighting the mathematical practices listed above. Submissions could be any of the following:

- Lesson plans
- Classroom management tips
- Books reviews
- Classroom games
- Reviews of recently adopted resources
- Good problems for classroom use

Email your submissions to our Bulletin editor: wilcojen@usd437.net

Acceptable formats for submissions: Microsoft Word document, Google doc, or PDF.

Call for Nominations!!!

The KATM Board is currently taking nominations to fill the following positions in the upcoming Board election. We are looking for educators that are interested in taking a leadership role in the field of Math Education throughout the State of Kansas. You can nominate yourself or someone that you know that has demonstrated a passion for advancing math in our state as well as someone that has a lot to offer in the way of supporting teachers. Please email Fred Hollingshead, Past President (hollingsheadf@usd450.net), with nominations and contact info of the nominee or fill out the online nomination form found at katm.org. Regular members in good standing are eligible for positions on the KATM Board. Nominations need to be completed by February 8th. Elections will be held online at www.katm.org starting March 1st and ending March 7th. A notice will be sent to remind you to vote.

Positions available for the upcoming election:

President-elect * 4-year term

The president-elect will serve for one year before then becoming president for a year, and then past-president for two years. The president-elect will assume the duties of president when needed. As president, the elected individual will preside over all KATM events and business meetings. The president will conduct the business of KATM as directed by the Executive Board and will represent KATM at a variety of functions, meetings, and conferences. The president is responsible for the overall functioning of the organization with assistance from the officers and Board members. As the past-president taking office in even-numbered years, this position will serve as the community relations representative for 2 years. This person shall be responsible for assuring communication between the Association and legislative, executive, and administrative branches of the government of Kansas.

Vice President – Elementary * 2-year term

The vice president for college will attend all meetings and conferences and will assist the president in conducting the business of KATM. Each vice president will also encourage membership and will promote issues of special interest to their level represented in addition to serving on various committees as assigned. The person elected to this position will act as a liaison to college teachers.

Vice President – High School * 2-year term

The vice president for middle school will attend all meetings and conferences and will assist the president in conducting the business of KATM. Each vice president will also encourage membership and will promote issues of special interest to their level represented in addition to serving on various committees as assigned. The person elected to this position will act as a liaison to middle school teachers.

We also have openings for zone coordinators in Zone 2 (North Central) and Zone 4. (South East)

Do you like what you find in this Bulletin? Would you like to receive more Bulletins, as well as other benefits?

Consider becoming a member of KATM.

For just $15 a year, you can become a member of KATM and have the Bulletin e-mailed to you as soon as it becomes available. KATM publishes 4 Bulletins a year. In addition, as a KATM member, you can apply for two different $1000 scholarship.

Current members—-refer three new members and you get one free year of membership!

Join us today!!! Complete the form below and send it with your check payable to

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Are you a member of NCTM? Yes___ No___

Position: (Cirlce only one)

Parent
Teacher:: Level(s)________
Dept. Chair
Supervisor
Other

Referred by: _________________________________

____

KANSAS ASSOCIATION MEMBERSHIPS

Individual Membership: $15/yr. ___
Three Years: $40 ___
Student Membership: $ 5/yr. ___
Institutional Membership: $25/yr. ___
Retired Teacher Membership: $ 5/yr. ___
First Year Teacher Membership:$5/yr. ___
Spousal Membership: $ 5/yr. ___
(open to spouses of current members who hold a regular Individual Membership in KATM)

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