Just Say Yes to Early Algebra!

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Researchers find that these classroom activities and instructional strategies support the development of third-grade students' algebraic thinking.

Mathematics educators have argued for some time that elementary school students are capable of engaging in algebraic thinking and should be provided with rich opportunities to do so. Recent initiatives like the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) have taken up this call by reiterating the place of early algebra in children's mathematics education, beginning in kindergarten. Some might argue that early algebra instruction represents a significant shift away from arithmetic-focused content that has typically been taught in the elementary grades. To that extent, it is fair to ask, "Does early algebra matter?" That is, will teaching children to think algebraically in the elementary grades have an impact on their algebra understanding in ways that will potentially make them more mathematically successful in middle school and beyond?

Plenty of evidence certainly exists that elementary school students can think algebraically about particular concepts. For example, we know that students can develop a relational understanding of the equal sign (Carpenter, Franke, and Levi 2003; Falkner, Levi, and Carpenter 1999); generalize important arithmetic relationships such as the Commutative Property of Multiplication (Bastable and Schifter 2008; Schifter 1999); and use representations such as tables, graphs, and variable notation to describe functional relationships (Blanton 2008; Carraher et al. 2006). However, it is also important to know how children think algebraically *across* a comprehensive set of algebraic concepts in content domains that, at first glance, might not seem deeply connected.

In this article, we share findings from a research project whose goal is to study the impact of a comprehensive early algebra curricular experience on elementary school students' algebraic thinking within a range of domains including generalized arithmetic, equivalence relations, functional thinking, variables, and proportional reasoning. We focus here on the performance of third-grade students who participated in our early algebra intervention on a written assessment administered before and after instruction. We also discuss the strategies these students used to solve particular tasks and provide examples of the classroom activities and instructional strategies that we think supported the growth we saw in students' algebraic thinking.

We believe the research presented here paints a compelling picture regarding the potential for elementary school students to successfully engage with a range of early algebraic concepts, and we believe that sharing this with educators—who are increasingly expected to develop children's algebraic reasoning (CCSSI 2010)—is important.

Our early algebra intervention

Two third-grade classrooms with a combined total of thirty-nine students participated in our intervention. Students' regular mathematics curriculum contained little algebra. Our instructional sequence consisted of approximately twenty one-hour early algebra lessons throughout the school year that took the place of students' regularly scheduled mathematics instruction for that day. Each lesson began with small-group discussions of previously taught concepts, and then new concepts were introduced through small-group problem solving and whole-class discussion. One member of our research team, a former elementary school teacher, taught all the lessons.

In this article, we discuss students' responses to a representative sample of items from the preassessment and postassessment (see Blanton et al. 2015 for a more thorough presentation of assessment results) and the nature of the instruction that supported their learning.

Results

How do you think your own students would respond to a representative sample of assessment items (see **fig. 1**)? Students who participated in our instruction made significant gains in their abilities to view the equal sign as a relational symbol, identify arithmetic properties (e.g., the Commutative Property of Addition), write variable expressions to represent unknown quantities, and generalize and express functional relationships.

In addition to whether students responded correctly to each assessment item, we were also interested in the types of strategies they used and whether the strategies that students used at the end of our instruction reflected more algebraic ways of thinking than those they had used before our instruction. We found that students who had the opportunity to engage in early algebraic thinking throughout the course of the school year tended to approach the assessment items more algebraically and were more apt to "look for and make use of structure," one of the Common Core's (CCSSI 2010) Standards for Mathematical Practice (SMP 7, http://www.corestandards.org/Math/Practice/). In what follows, we discuss the strategies that students used to solve the items (see **fig. 1**) and highlight the structural thinking that we observed.

How did students "look for and make use of structure"?

Equality

The fact that many students view the equal sign as an operational symbol meaning "give the answer" has been well documented (e.g., Behr, Erlwanger, and Nichols 1980; Carpenter, Franke, and Levi 2003). We likewise found that the vast majority of students were unsuccessful with the equality items during pretesting (see **fig. 1a**) and gave responses indicating they viewed the equal sign operationally by placing a 10 or 14 in the blank in 7 + 3 = ___ + 4 or by stating that 57 + 22 = 58 + 21 is false because, for example, "57 + 22 = 79, not 58." However, students clearly came to view the equal sign as a relational symbol over the course of our instructional intervention (see **fig. 1a**). For many of these students, growing knowledge of the equal sign as meaning "the same value as" in arithmetic and algebraic equations led

FIGURE 1

Researchers found that with instruction, students made significant gains in their abilities to view the equal sign as a relational symbol, identify arithmetic properties, write variable expressions to represent unknown quantities, identify recursive patterns, and generalize and express functional relationships in both words and variables.

Student performance on a representative sample of assessment items

Assessment item	Percentage of students who provided correct responses

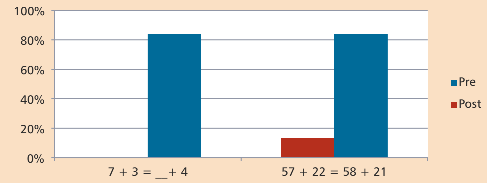
(a) Equality

Fill in the blank with the value that makes the following number sentence true. How did you get your answer?

$$7 + 3 = \_\_\_\_ + 4$$

Circle True or False and explain your choice.

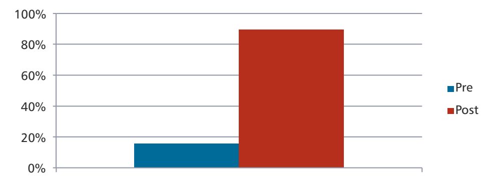
57 + 22 = 58 + 21 True False



(b) Generalized arithmetic

Circle True or False and explain your choice.

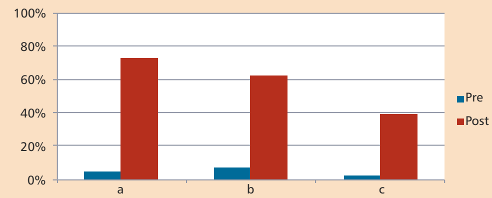
39 + 121 = 121 + 39 True False



(c) Writing variable expressions

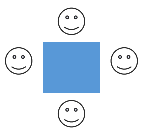
Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don't know how many. Angela also has 8 pennies in her hand.

1. How would you describe the number of pennies Tim has?
2. How would you describe the total number of pennies Angela has?
3. Angela and Tim combine all their pennies to buy some candy. How would you describe the total number of pennies they have?

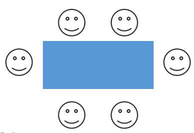


(d) Functional thinking

Brady is having his friends over for a birthday party. He wants to make sure he has a seat for everyone. He has square tables. He can seat 4 people at one square table in this way:

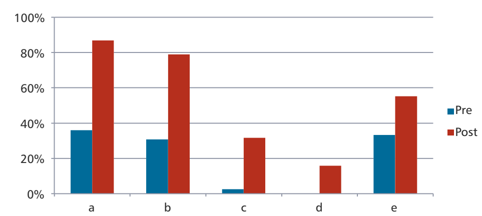


If he joins another square table to the first one, he can seat 6 people:



1. If Brady keeps joining square tables in this way, how many people can sit at 3 tables? At 4 tables? At 5 tables? Record your responses in the table to the right and fill in any missing information.
2. Do you see any patterns in the table? Describe them.
3. Find a rule that describes the relationship between the number of tables and the number of people who can sit at the tables. Describe your rule in words.
4. Describe your relationship using variables. What do your variables represent?
5. If Brady has 10 tables, how many people can he seat? Show how you got your answer.

No. of tables	No. of people
1	
2	
3	
4	
5	
6	
7	



MANIPULATIVES ARE USEFUL TOOLS THAT HELP PROMOTE THE IDENTIFICATION OF RELATIONSHIPS AND MATHEMATICAL STRUCTURE.

them to compute sums on both sides of these equations to find the missing value in 7 + 3 = ___ + 4 or to determine the validity of 57 + 22 = 58 + 21. However, many of them went a step further and developed the ability to view these equations structurally and successfully solve these items without using computation. By posttest, 16 percent of students gave an explanation indicating they solved 7 + 3 = ___ + 4 by attending to structure (e.g., "Four is one more than three, so the blank must be one less than seven"), and 29 percent of students gave an explanation indicating they solved 57 + 22 = 58 + 21 by attending to structure (e.g., "Fifty-eight is one more than fifty-seven, and twenty-two is one more than twenty-one, so it's true").

Generalized arithmetic

One of the core areas of early algebra is generalized arithmetic, whereby students deepen their arithmetic understanding by noticing and representing regularity and structure in their operations on numbers. When asked whether 39 + 121 = 121 + 39 was true or false, none of the students who responded correctly during the pretest gave an explanation that relied on the equation's underlying structure. They tended, rather, to compute the sums separately on each side of the equal sign and find 160 = 160. At the posttest (see **fig. 1b**), however, 66 percent of students provided this type of explanation (e.g., "True, because 121 + 39 is just 39 + 121 in reverse").

Writing variable expressions

Students who confront an unknown quantity are often uncomfortable with this ambiguity and want to assign a specific value rather than use a variable (Carraher, Schliemann, and Schwartz 2008). Likewise, we found that students were unable to represent unknown quantities symbolically at pretest time (see **fig. 1c**) and that those who responded to this item did so by choosing a numerical value to represent Tim's number of pennies (e.g., "Tim has ten pennies"), even though the item specifically stated that the quantity is unknown.

It is often assumed that young students are not "developmentally ready" to work with variables and should instead work exclusively with concrete representations. Our findings suggest, however, that students who are provided with the appropriate experiences *can* engage quite successfully with symbolic representations. In response to question 1 in **figure 1c**, no student assigned a specific numerical value to the unknown quantity at posttest and, in fact, 74 percent used a variable to represent the quantity (e.g., "Tim has n pennies").

Further, students' posttest responses to questions 2 and 3 highlight their abilities to attend to mathematical structure and treat expressions as single objects. We found that 63 percent of students were able to use variable notation to represent Angela's number of pennies in a way that connected to their representation of Tim's number of pennies in question 1. In other words, these students understood that if n represented Tim's number of pennies, then Angela's number of pennies could be best represented by $n + 8$. Similarly, 39 percent of students provided a representation in part c that related to those in questions 1 and 2. For example, if students represented Tim's number of pennies as n in question 1, then these students might represent the combined number of pennies for Tim and Angela as $n + n + 8$ in question 3. We believe this indicates that these students were using variables with understanding and were thinking structurally by building on previously established expressions.

Functional thinking

Functional thinking involves reasoning about and expressing how two quantities vary in relation to each other (Blanton 2008). This algebraic domain unfortunately often receives little attention in the elementary grades (Blanton and Kaput 2011) even though it is a significant part of CCSSM in later grades. We found, however, that with instruction, young students can learn to recognize and express functional relationships. As **figure 1d** shows, students made gains in their abilities to complete function tables, identify recursive patterns, generalize functional relationships, and represent these generalizations in both words and variables. See Isler and her colleagues' (2015) detailed account of student performance on this assessment item and the classroom activities that

FIGURE 2

Students explored even and odd numbers by representing numbers with cubes.

How many pairs?

Use your cubes to complete the following table for the given numbers.

No.	No. of pairs created	No. of leftover cubes
3		
4		
5		
6		
7		

What do you notice? What kinds of numbers have no cubes left after all pairs are made? What kinds of numbers have a cube left? Write a sentence to describe each of your observations.

contributed to the development of students' functional thinking.

How did students' algebraic thinking develop?

How did students—during the course of one school year—develop such sophisticated ways of thinking about a wide range of algebraic concepts? While focusing on the algebraic domains mentioned above, students were also asked to engage in four algebraic thinking practices that are central to the discipline and align to a great extent with the Common Core's SMP (CCSSI 2010). In what follows, we discuss each of these practices and use students' work exploring even and odd numbers as examples to illustrate what this thinking looked like in our classrooms and what it might look like in yours.

First, students were routinely posed tasks that encouraged them to *generalize mathematical relationships and structure*. This type of thinking occurs when students notice relationships or structure in arithmetic operations, expressions, equations, or function data that can be generalized beyond the given cases. For example, students in our classrooms were asked to explore representing numbers with cubes so that they might come to identify two types of numbers—even and odd (see **fig. 2**).

Students noted that the numbers in the table

FIGURE 3

Students explored sums of even and odd numbers. The researchers discovered that manipulatives are useful tools in promoting the identification of relationships and mathematical structure.

1. Jesse is adding two even numbers. Do you think his answer will be an even number or an odd number?
2. Jesse is adding two odd numbers. Do you think his answer will be an even number or an odd number?
3. Jesse is adding an even number and an odd number. Do you think his answer will be even or odd?

alternated between having zero and one "leftover." They concluded that even numbers—the ones with zero leftover cubes—always have a "buddy." They also noticed that even numbers, when divided into two rows of cubes, form a rectangle; whereas odd numbers always have one cube sticking out by itself.

After exploring properties of even and odd numbers, students were asked to think about sums of even and odd numbers by working on the task shown in **figure 3**. As students explored both representing sums of numbers with cubes and computing specific sums of evens and odds, they began to notice important structures in even and odd numbers and their sums. We found that in the context of generalized arithmetic in particular, manipulatives are useful tools that help promote the identification of relationships and mathematical structure.

Once students had recognized mathematical relationships, we often asked them to *represent generalizations*. Students can use various notational systems—words, symbols, tables, graphs, and pictures—to represent their generalizations. In the case of the questions we posed (see **fig. 3**), our students used words to express conjectures, such as "An even number plus an even number is an even number" and "An even number plus an odd number is an odd number." In a few years, these students should be able to use symbolic notation to express an even number as $2n$ and an odd number as $2m + 1$ (for any integers n and m). Natural language, however, can be a useful scaffold for developing an understanding of symbolic notation.

Students in our intervention were also asked to *justify generalizations*. When asked to justify a generalization they have expressed verbally or symbolically, students often begin by offering numerical examples. We found this to be

NOTICE THAT ASKING SUCH QUESTIONS AS "DO YOU THINK THIS WILL ALWAYS WORK?" AND "*WHY* DOES THIS ALWAYS WORK?"—AS WELL AS REFERRING STUDENTS BACK TO THEIR PREVIOUS "DEFINITIONS"—HELPED STUDENTS MOVE BEYOND EXAMPLES-BASED REASONING.

true in the case of students' explorations with even and odd numbers, with students saying, for example, "I know an even plus an even is an even because 2 + 4 = 6." It is important, however, that students learn to appreciate the limitations of "justification by example" and move toward making general arguments. Think about how our teacher encouraged this shift in students' thinking by considering the following excerpt of classroom dialogue:

Student 1: We could say that when you add an even number plus an even number, the sum will be even.

Teacher: I love that. Now, do you think this will always work? Have we shown or tried enough examples to be sure that this will always work?

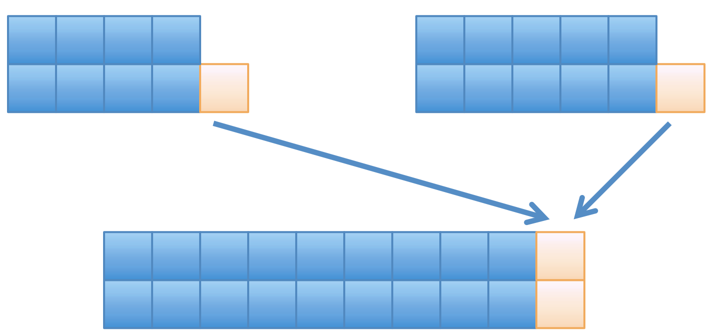
Student 1: No, we should probably try a few more. [*Students add more even numbers and write sums.*]

Teacher: So, how are we feeling? Do you still feel that an even plus an even will always be even?

Student 2: Yes, because I tried a bunch of examples and it works for all of them.

FIGURE 4

This student used representation-based reasoning to justify why the sum of two odd numbers is an even number.



Teacher: Great. I agree. I think that when we add an even plus an even, the sum will always be even. But, *why*? *Why* does this always work?

[*Students give more examples.*]

Teacher: Yes, I agree. You have shown me a great number of examples, but *why*? What did we learn about even numbers when we were exploring a little while ago?

Student 3: Even numbers always have pairs!

Teacher: OK, so could that help us answer why an even plus an even is an even?

Student 3: Yes, because when we add even numbers, we don't ever start with any leftovers, everyone has a pair; so we can add them together, and everyone will always have a pair.

Notice that asking such questions as "Do you think this will always work?" and "*Why* does this always work?"—as well as referring students back to their previous "definitions" of even and odd numbers—helped students move beyond examples-based reasoning.

Talking about numbers in general can be difficult for children when they are accustomed to working with specific values. Sometimes, however, specific examples can be used in such a way that students' justifications do not depend on the specific numbers used. Consider, for example, how the following student used cubes to justify that the sum of two odd numbers is an even number:

> I did it with blocks. So, I took 9 blocks, and I added it to 11. If you look at the blocks alone, 9 and 11, they each have a leftover, but when you put them together, their leftovers get paired up, so you have an even number. [*See* **fig. 4**.]

Notice that although this student's justification used nine blocks and eleven blocks, there is nothing special about these specific numbers. Any odd numbers could have been chosen to make the argument. Furthermore, the student did not need to calculate in the process of justifying the generalization. This type of justification is sometimes referred to as "representation-based" reasoning (Russell, Schifter, and Bastable 2011) because it relies on the use of a physical or visual representation as a bridge to a general argument. A good strategy is to question students about the specific examples they choose—“Did you have to count those cubes?” or “Does it only work for your example?”—to encourage them to engage in representation-based reasoning and begin to appreciate the power of general arguments.

Finally, students in our classroom were often encouraged to *reason with generalizations*. This occurs when students make use of generalizations to solve problems. Students often do this naturally, without being asked to do so and without explicitly thinking about the generalizations they are using. For example, when asked whether the sum of three odd numbers would be even or odd, our students were often able to build on the already-established generalization that the sum of two odd numbers is an even number. One student explained, for example, that two odd numbers equal an even number and—

> if you have an even number, it is all paired up. If you add that to an odd number, which has a leftover, you can never get rid of the leftover. It still has nothing to pair with, so your answer will always be odd.

Part of engaging students in thinking algebraically involves posing tasks that encourage the use of a particular generalization and then helping students make the taken-for-granted generalization explicit.

Can young students be successful in algebra?

Overall, our study’s results reveal that elementary school students who experience a comprehensive and sustained early algebra education—that is, across multiple algebraic domains and spanning an entire school year—can successfully engage with a variety of algebraic content that is often reserved until middle school or later. The ability to think structurally is an important aspect of algebraic thinking

(Kieran 2007), and we found that third-grade students in our study were capable of this type of reasoning. Keeping these results in mind, we encourage you to work with your students in the algebraic domains discussed here and engage them in the important algebraic thinking practices of generalizing mathematical relationships and structure, expressing generalizations, justifying generalizations, and reasoning with generalizations.

Common Core Connections
SMP 3
SMP 5
SMP 7

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