During a lesson on ratios involving percentages of paint, four research-based criteria are used to evaluate students' mistakes. The takeaway is that painting all mistakes with the same brush can also be a blunder.

Examining Mistakes to Shift Student Thinking

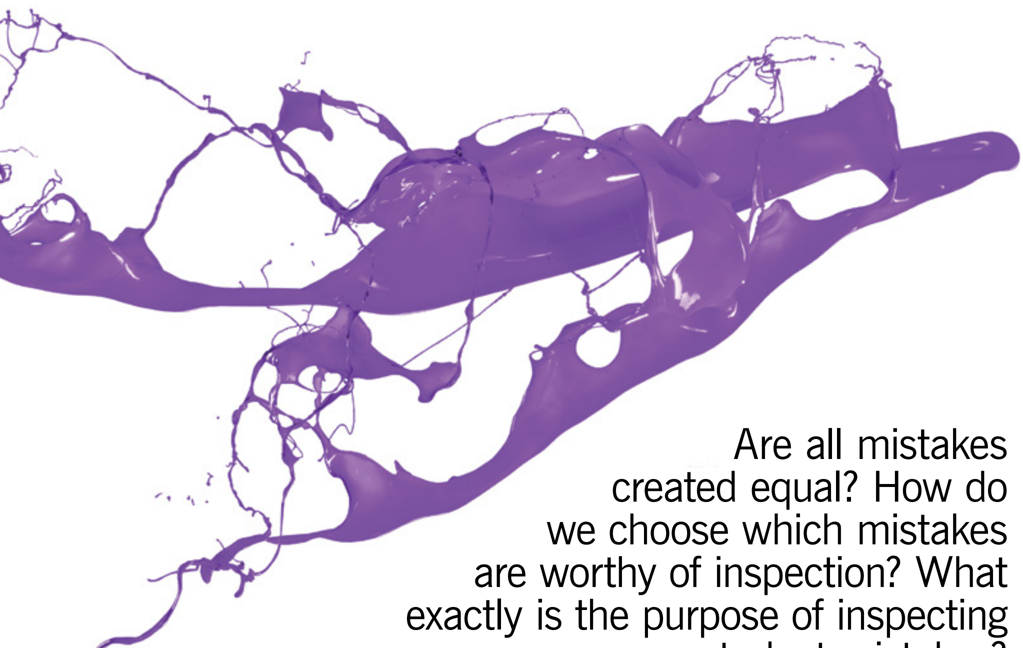
James C. Willingham, Jeremy F. Strayer, Angela T. Barlow, and Alyson E. Lischka

Middle-grades teachers and students can have different perspectives on the value of discussing students' mathematical mistakes, despite various classroom evidence that such discussions can help foster strong conceptual understanding (Boaler 2016). Some teachers consider student mistakes to be an opportunity to correct errors in individual student thinking. Others view the public inspection of mistakes as an opportunity for all students in the classroom to learn.

Although both of these perspectives take students' learning into account, our students often regard their own mistakes in a very personal manner. They see mistakes as flaws for which their teachers will judge them.

Because of the variety of perceptions regarding the value of mathematical mistakes, it is imperative that teachers consider how to leverage mistakes during classroom instruction. Are all mistakes created equal? How do we choose which mistakes are worthy of inspection? What exactly is the purpose of inspecting student mistakes? Is it simply to correct faulty answers? How should we use these mistakes during instruction? Reflecting on our teaching practice in light of these questions can lead to helpful insights. It is our hope that by considering the pedagogical quality of the mistakes examined in the classroom, both students and teachers will deepen their understanding of the value of mathematical mistakes for learning.

We begin with a statement by former NCTM President Linda Gojak: "Helping students to learn from their mathematical mistakes can give us insight into their misconceptions and, depending on our instructional reactions, can enable them to develop deeper understanding of the mathematics they are learning" (Gojak 2013, para. 4). A number of classroom tools are available that take advantage of this powerful idea, including setting up classroom norms that value mistakes (Boaler 2016); planning and selecting tasks to elicit mistakes (Bray 2013); helping students focus on and discuss mistakes in meaningful ways (Pace and Ortiz 2016); and assessing and designing responsive instruction based on student mistakes (Barlow et al. 2016). The purpose of this article is to add to this set of tools a list of criteria for determining *which* student mistakes are worthy of class examination. As we discuss the criteria, we offer insight into *why* certain mistakes are worthy of inspection and *how* teachers might leverage their examination to shift students' mathematical thinking forward within the context of a specific mathematics lesson. Our criteria are aimed at supporting the learning of *each and every* student, not just those who made the initial mistake.



Are all mistakes created equal? How do we choose which mistakes are worthy of inspection? What exactly is the purpose of inspecting student mistakes?

SELECTING MISTAKES

Merely drawing attention to and correcting errors in calculation does not typically have a large pedagogical payoff for a deep learning of mathematics. Therefore, what kinds of mistakes might be worth inspecting so that students' mathematical thinking is moved forward in the learning process? Some mathematics education literature (Barlow et al. 2016; Kilpatrick, Swafford, and Findell 2002; NCTM 2014) provides guidance for deciding which student mathematical mistakes might be worthy of closer inspection. On the basis of this literature and our own experiences inspecting mistakes in classrooms characterized by students sharing their mathematical thinking and discussing their different solutions to nonroutine, rich mathematics problems, we offer the following criteria for choosing inspection-worthy mistakes:

1. The mistake is closely aligned with the mathematical goals of the lesson.
2. The mistake provides powerful insight into students' conceptual understanding, fluency with procedures, or competence in selecting strategies for problem solving.
3. The mistake aligns well with the class's general progress toward solving the problem.
4. The mistake offers a viable answer that may be contrary to the class's accepted solution or solution strategies because of hidden assumptions about the problem (e.g., the student interpreted the problem differently than intended).

If a student mistake meets one or more of these criteria, it is probably a good candidate for whole-class inspection.

The remainder of this article describes how we applied these criteria to select mistakes for class inspection during a lesson with preservice teachers addressing a task designed for sixth- and seventh-grade students. The task focused on understanding ratios and percentages, and it elicited mistakes in the preservice teachers' work that were similar to those that might be expected from students at this grade level. For this lesson, we used a problem involving mixing paint, presented in the next section, as the central problem-solving task.

THE PURPLE PAINT PROBLEM

The mathematical goal we sought to achieve with the Purple Paint problem was to have students track multiple part-whole relationships in a complex multistep problem and use these relationships appropriately to reason with ratios and percentages. The Purple Paint Problem follows.

> Katie wants to paint her bedroom a special shade of purple made up of equal amounts of pink and powder-blue paint. To make the pink paint for this mixture, she combines one part red paint with one part white paint. To make the powder-blue paint, she mixes three parts blue paint with one part white paint. Finally, to make the purple paint, she mixes equal parts of the pink and powder-blue paints.
>
> If Katie needs two gallons of purple paint to finish her bedroom, how many quarts of blue, red, and white paint should she buy? What percentage of the purple paint comes from blue paint? What percentage comes from white paint?
>
> Use diagrams, symbols, and words to justify that your answers are correct.

Fig. 1 This sample of correct student work for the Purple Paint problem met the lesson's expectations.

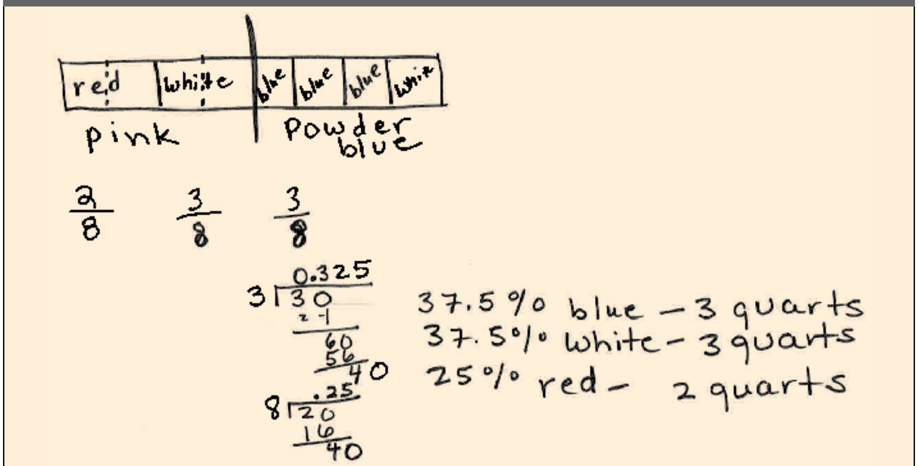
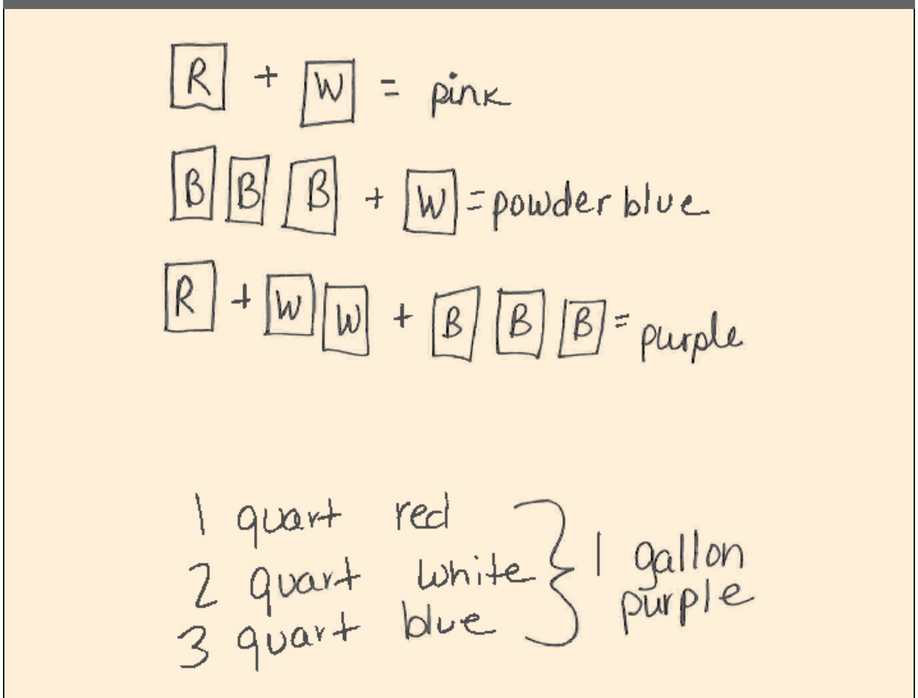


Fig. 2 This mistake demonstrated an error in conceptual understanding.



A sample of correct student work for this problem, indicating progress toward achieving our lesson goals, is included in **figure 1**.

An equally important goal in our implementation of the Purple Paint problem was to allow each and every student in the classroom access to the rich mathematics embedded in the task. This access is required to produce mathematically meaningful mistakes that address the important concepts of the problem and support students in generating new mathematical understanding. For English language learners and other students who might struggle with reading or processing the task, we offer two suggestions:

1. In addition to posting the problem in a place where all students can read it, plan to read the problem aloud, physically demonstrate the actions involved in the

problem, and offer appropriate translations. To ensure that students understand the intent of the task, ask them to revoice the problem while working in pairs, and then select a student to explain to the whole class what he or she thinks the problem is asking.

2. Consider the technique of Delaying the Questions (Barlow et al. 2017) to provide students an opportunity to make sense of the problem's underlying relationships before introducing the specific questions that they will be addressing. The Purple Paint problem is ideal for this technique because it contains a problem stem that introduces the relationships between each of the paint mixtures prior to the problem's questions. When students are allowed time to consider the ratio relationships ingrained in the problem stem, they will be much better prepared to apply their understanding of these relationships to the remainder of the problem.

Fig. 3 This work demonstrated errors in procedural fluency based on a previously examined conceptual error.

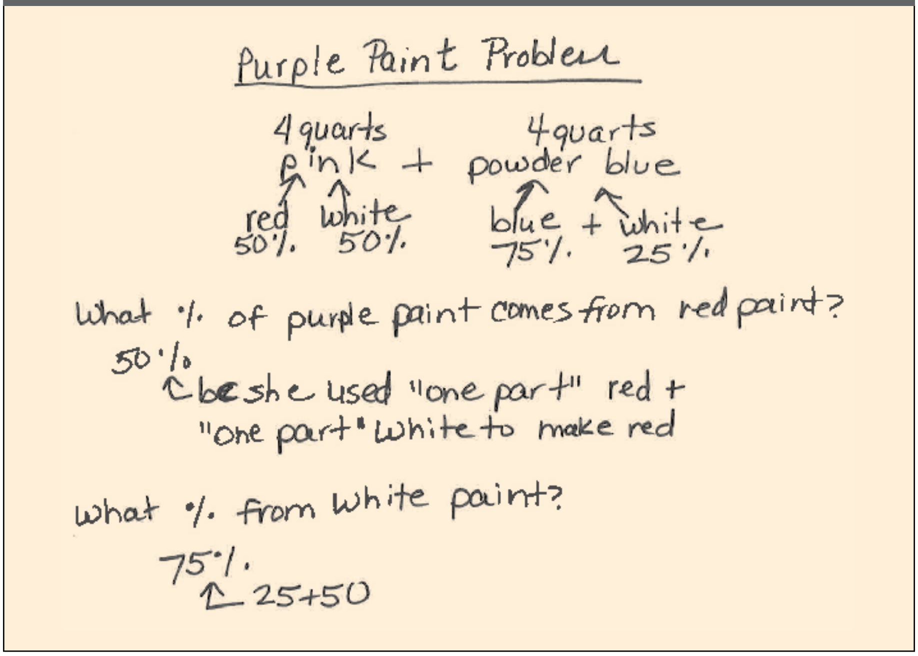
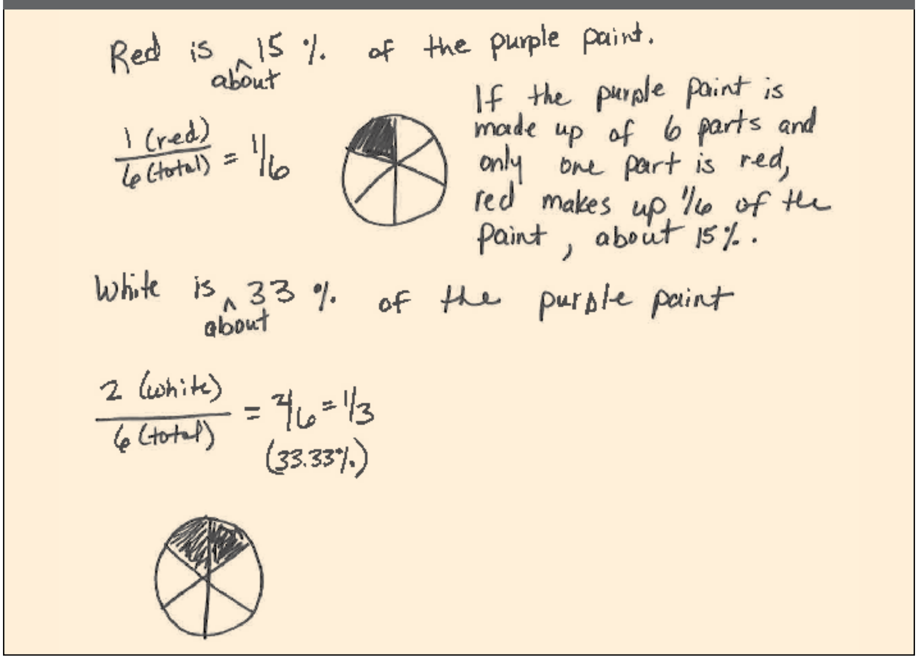


Fig. 4 In this sample of student work, percentages were correctly determined based on a previous part-to-whole error.



SELECTING CONCEPTUAL MISTAKES ALIGNED WITH THE MATHEMATICAL GOAL OF THE LESSON

After introducing the problem, the teacher, who is also the lead author of this article, asked students to take a minute or so to think privately about how they might solve it before moving into groups of four to negotiate a solution. As students worked, the teacher circulated among them, observed their approaches, asked advancing questions, and considered which mistakes might produce an impactful whole-group discussion. The first mistake selected for whole-class inspection (see **fig. 2**) was chosen because it aligned with criteria 1 and 2. Specifically, the work focused on the central ideas of the lesson goal and contained a conceptual misunderstanding of the part-to-part relationships within the larger whole in the problem. Although the students who produced this work accurately represented the relationships of equal-sized parts in the smaller wholes (the mixtures of pink and powder-blue paints), they did not take into account how this relationship impacted the composition of

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Students were able to move their thinking forward as they talked about the percentages, when referring to different wholes in the context of paint.

the larger whole (the mixture of purple paint).

As groups of students began to complete the task, the teacher collected the sample in **figure 2** and displayed it via the document camera to be inspected by the class. Small groups were asked to compare the work with their own findings, focusing on how they had represented each of the mixtures in the problem. As students discussed this representation, several important ideas emerged and were shared with the class. One group agreed that the representations for the pink and powder-blue paints were correct by themselves but suggested that they had to be adjusted to show equal amounts of these mixtures in the purple mixture. A second group added that this would require either doubling the number of units used in the pink paint or halving the number used in the blue. Another group observed that the units in the representation were quarts of paint, and because the final amount of paint called for in the problem was equivalent to eight quarts, only doubling the number of units in the pink paint would actually give this amount. Through this process, students were able to shift their thinking to understand more deeply the idea of scaling a quantity based on its part-to-part relationship, a critical idea in solving problems with ratio reasoning.

SELECTING FLUENCY MISTAKES ALIGNED WITH CLASS PROGRESS AND LESSON GOALS

The teacher selected the second mistake for inspection (see **fig. 3**) because it aligned with criteria 1, 2, and 3. This work contained an error in procedural fluency related to the conceptual error in **figure 2**. Although the students who produced this work showed evidence of reasoning correctly on the first part of the problem, they calculated percentages for the subparts and assumed that those transferred directly to the larger whole. This procedural mistake was not only in line with the lesson's goal but also aligned well with the class's general progress toward solving the problem. In addition, this particular mistake was pervasive throughout the work of several groups.

The teacher directed students to consider this work in their small groups and then report on what they noticed. The resulting whole-class discussion focused on the meaning of *percentage*. Several students questioned the validity of their peers' claims that the white paint comprised 75 percent of all the paint when the work could also be interpreted as showing that the white paint made up 3 out of 8 parts of the whole. Other students noted that it was unreasonable to say that red paint comprised 50 percent of the paint and at the same time claim that white paint made up 75 percent of the paint because this sum was greater than 100 percent without accounting for any of the blue paint.

The benefit of selecting a mistake based on criterion 3 was evident when students reflected on how the percentage error related to the previous discussion of the size of the parts compared with the whole (see the discussion of **fig. 2**). Indeed, students were able to move their thinking forward as they talked about the percentages, when referring to different wholes in the contexts of the pink paint, powder-blue paint, and purple paint.

SELECTING MISTAKES THAT ARE POTENTIALLY CONTRARY TO AN ACCEPTED SOLUTION

Sometimes after a class has come to some conclusions about the solution to a problem, it can be beneficial to challenge their thinking. It was for this purpose that the teacher selected a final sample of student work (see **fig. 4**) on the Purple Paint problem according to criteria 1, 2, and 4. The students who produced this work showed a solution that contained dramatically different percentages from those that had been reported by most of the class. It appears that these students used reasoning similar to that of the students who presented the work in **figure 2** on the ratio portion of the problem, believing the purple paint mixture to comprise one part red paint, two parts white paint, and three parts blue paint. Although this reasoning was incorrect, the process they used to determine the percentages of the purple paint mixture was appropriate. These students eventually resolved this error on their own, but the teacher used the Get the Goof

Fig. 5 Some mistakes arose from a failure to read and represent the problem or solution carefully.

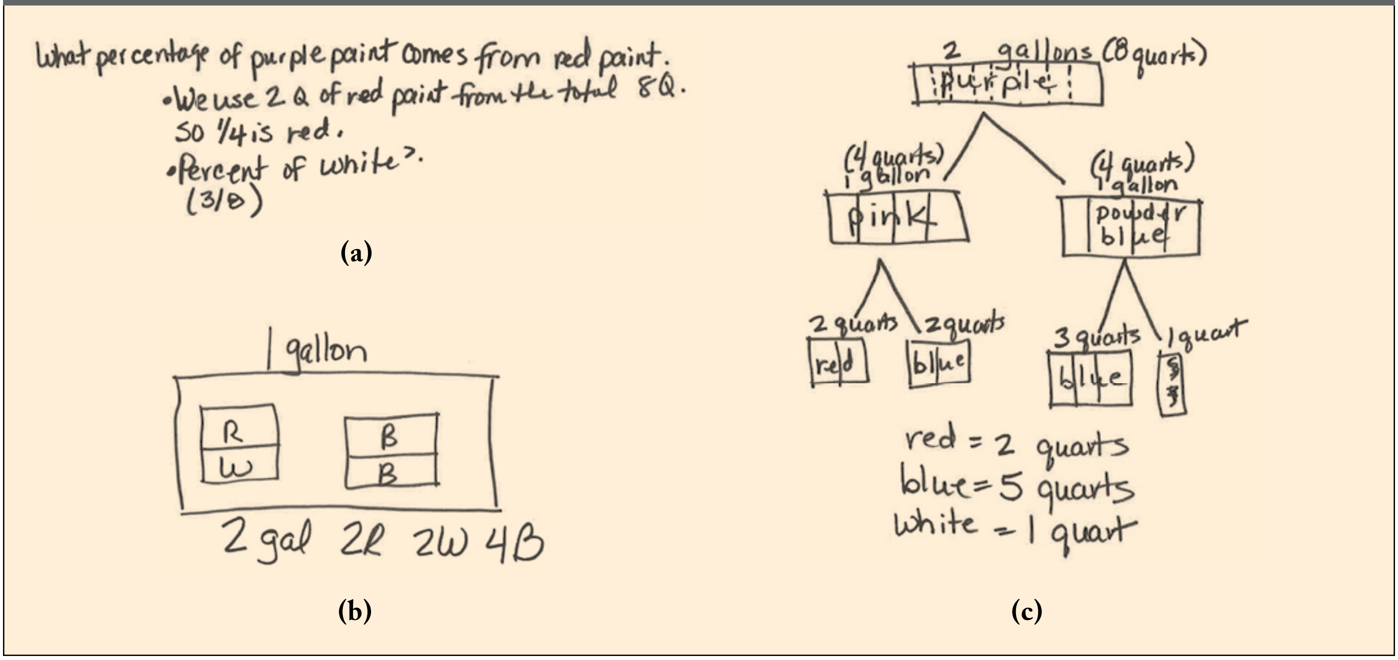
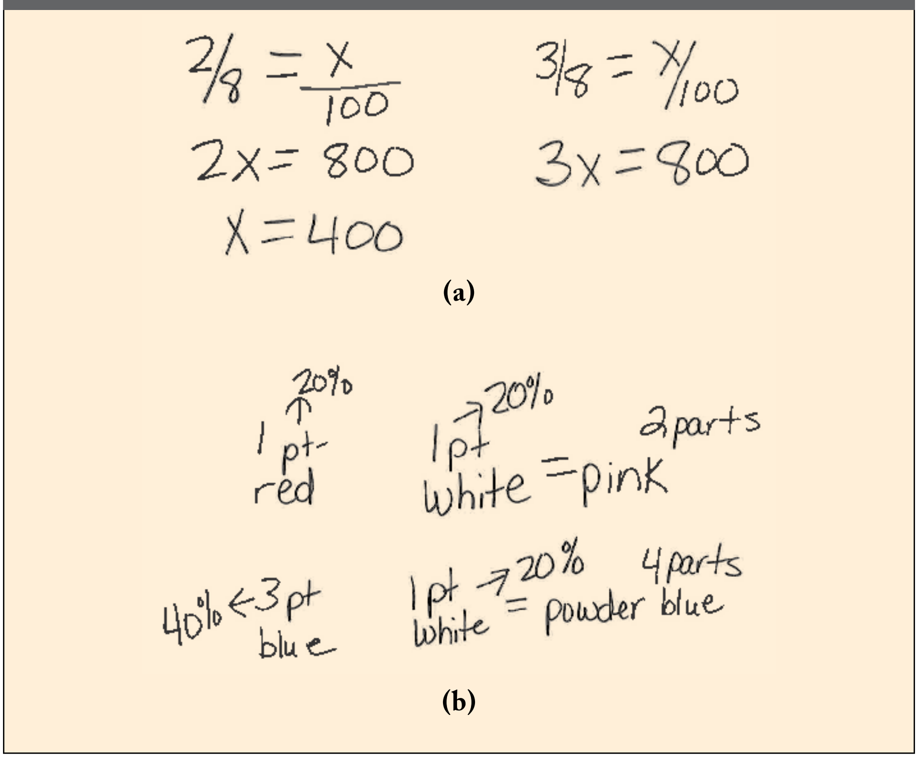


Fig. 6 Some mistakes involved an inappropriate or incorrect application of procedures.



strategy described by Pace and Ortiz (2016) to see whether other students could explain the mistake. Resolving the conflict that this mistake created provided an opportunity for the class to solidify their conceptual understanding according to the mathematical goals of the lesson.

Now that we have considered some examples of inspection-worthy mistakes, we find it helpful to distinguish them from those that are not inspection-worthy mistakes. We close the article with a brief discussion of mistakes that were made but were not chosen for class inspection during the Purple Paint problem.

DETERMINING WHICH TYPES OF MISTAKES WILL NOT BE INSPECTED AS A CLASS

If mistakes are minor, isolated cases and do not align well with the mathematical goals of the lesson, it can be difficult for a teacher to use them in a whole-class inspection with the goal of helping each and every student learn mathematics deeply. Often these mistakes are better addressed with brief one-on-one interactions with the student that do not detract from the lesson's primary mathematical goals. These kinds of mistakes occurred while students worked with the Purple Paint problem:

- Mistakes in calculation, such as errors in performing long division.

- Mistakes involving a missing piece of specific knowledge, for example, not knowing that there are four quarts in one gallon.
- Mistakes arising from a failure to read or represent the problem or its solution carefully (see **fig. 5**).
- Mistakes involving an inappropriate or incorrect application of a procedure (see **fig. 6**).

In **figure 5a**, we see the work of a group of students who correctly determined that 3/8 of the purple paint was white but incorrectly stated that 3/8 was also the percentage of white paint. In **figure 5b**, we see the work of students who treated the powder-blue paint as if it were pure blue paint and failed to include any white paint in their representation of the powder-blue portion of the purple paint mixture. Finally, in **figure 5c**, we see a representation in which the students incorrectly labeled two quarts of the pink paint blue instead of white. In these cases, the teacher can help students pay attention to their mistakes individually and move toward a solution that is aligned with the lesson goals by asking these one-on-one questions of each group: "What do you mean that the percentage of white paint is 3/8? Is there another way you can state this?" "Have you represented the white paint in the powder-blue paint? Why might this be important as you answer the problem?" and "I see that you have the pink paint made up of red and blue paint. Is that what you intended?"

These personal interactions can be used to help shift thinking into mathematically productive areas and avoid the negative reactions that students sometimes experience because of mistakes of this nature.

Figure 6a displays work indicating that the students incorrectly carried out a procedure to determine the percentage value for a given fraction. **Figure 6b** displays work in which it is unclear how the students determined the percentages based on the number of parts identified. In each case, a teacher can help students individually move forward by asking such questions as "What do you mean by 400? Are you saying that two-eighths is 400 percent?" and "How do you know that the percentages you specified represent the number of parts of the whole you found? Do these percentages accurately represent what the problem states?"

Because the kinds of mistakes identified in this section hinder students from productively moving toward a solution that achieves the mathematical goals of the lesson, it is helpful for teachers to address them in brief one-to-one interactions with students, as described above. This enables students to spend their time wrestling with the big ideas aligned with the mathematical goals of the lesson. A prolonged whole-class inspection of these types of mistakes would probably not be a judicious use of class time because they do not meet the criteria identified for inspection-worthy mistakes. We are not claiming, however, that these mistakes are unimportant. Indeed, teachers may find these kinds of mistakes helpful for identifying the focus of future lessons, according to their students' needs.

Conducting appropriate class inspections of mistakes in student work can create pedagogically powerful moments in the classroom.

INSPECTING MISTAKES TO DEEPEN UNDERSTANDING

Conducting appropriate class inspections of mistakes in student work can create pedagogically powerful moments in the classroom. In this article, we present four criteria that teachers can use when deciding which mistakes to inspect in a whole-class setting so that students can shift their mathematical thinking and achieve deep mathematical understanding. By focusing on mistakes that meet these criteria, teachers can move the focus away from the fact that a mistake was made and toward the reasons why the mistake is mathematically meaningful for learning. It is our hope that as students gain expertise in examining meaningful mistakes, they will eventually regard this skill as one of the most important mathematical tools they have at their disposal when solving problems.

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James C. Willingham, willinjc@jmu.edu, is an assistant professor of mathematics education at James Madison University in Harrisonburg, Virginia. He is most interested in the development of mathematical knowledge for teaching and effective K–12 mathematics teaching practices. **Jeremy F. Strayer**, jeremy.strayer@mtsu.edu, is an associate professor of mathematics education at Middle Tennessee State University in Murfreesboro, Tennessee. He is interested in creating professional development opportunities to support K–16 teachers working to implement standards-based teaching practices. **Angela T. Barlow**, abarlow5@uca.edu, is the dean of the graduate school and director of sponsored programs at the University of Central Arkansas in Conway, Arkansas. Her research interests focus on supporting the instructional change process in elementary mathematics classrooms. **Alyson E. Lischka**, alyson.lischka@mtsu.edu, is an assistant professor of mathematics education at Middle Tennessee State University. She is interested in how beginning and practicing teachers learn to implement mathematical practices in their classrooms.

Let's Chat about Examining Mistakes

On Wednesday, April 18, 2018, at 9:00 p.m. EST, we will expand on "Examining Mistakes to Shift Student Thinking" (pp. 324–32), by James C. Willingham, Jeremy F. Strayer, Angela T. Barlow, and Alyson E. Lischka Join us at #MTMSchat.

We will also Storify the conversation for those who cannot join us live. Our monthly chats fall on the third Wednesday of the month.

Ed note: The April issues of *Mathematics Teacher*, *Mathematics Teaching in the Middle School,* and *Teaching Children Mathematics* contain a trio of articles that discuss how to leverage mistakes. Each article will be its respective journal's Twitter chat offering as well as the free online preview, available to NCTM members only. In *TCM*, "Inspection-Worthy Mistakes: Which? And Why?" by Angela T. Barlow, Lucy A. Watson, Amdeberhan A. Tessema, Alyson E. Lischka, and Jeremy F. Strayer, explores how carefully selecting and leveraging student errors for whole-class discussions can benefit the learning of all. *MT* contains "Making Room for Inspecting Mistakes," by Alyson E. Lischka, Natasha E. Gerstenschlager, D. Christopher Stephens, Angela T. Barlow, and Jeremy F. Strayer, which describes how selecting errors to discuss in class and trying three alternative lesson ideas can help move students toward deeper understanding. "Examining Mistakes to Shift Student Thinking," by James C. Willingham, Jeremy F. Strayer, Angela T. Barlow, and Alyson E. Lischka, is the *MTMS* offering.

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- Rethinking What Each and Every High School Student Needs Related to Algebra and Functions
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