Visual Reasoning Tools in Action

Double number lines, area models, and other diagrams power up students' ability to solve and make sense of various problems.

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The integration of appropriate tools and technology is an important guiding principle for school mathematics (NCTM 2014). Although we tend to focus on integrating technology in school mathematics, this article discusses tools that have consistently been an important part of mathematics teaching and learning.

Mathematics teachers have always encouraged their students to draw pictures or diagrams to make sense of and solve problems. Pólya (1945) includes drawing figures as a useful heuristic. The Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) identifies the strategic use of appropriate tools as one of the mathematical practices and emphasizes the use of pictures and diagrams as reasoning tools. Starting with the early elementary grades, CCSSM discusses students' solving of problems "by drawing." In later grades, such specific forms of diagrams as number lines, area models, tape diagrams, and double number lines are mentioned.

Although these diagrams may not be a common feature in U.S. mathematics curricula, they are common

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in many East Asian curricula. For example, Beckmann (2004) and Ng and Lee (2009) describe how strip (or bar) diagrams are used in the Singaporean elementary school curriculum. Watanabe, Takahashi, and Yoshida (2010) discuss how different visual representations, including tape diagrams and double number lines, are used in Japanese elementary mathematics textbooks. Murata (2008) articulates how a consistent use of a particular diagram helps Japanese students make sense of mathematics. One of the recommendations in *Improving Mathematical Problem Solving in Grades 4 through 8: A Practice Guide* (Woodward et al. 2012, p. 1) is to "teach students how to use visual representations." Thus, the integration of appropriate visual representations is an important step toward putting principles into action.

But how do middle-grades students use pictures and diagrams to make sense of and solve problems? This article illustrates the power of these visual reasoning tools by describing how Japanese sixth and seventh graders used a variety of pictures and diagrams to solve and make sense of problems. We will also discuss potential challenges and opportunities that these visual reasoning tools offer to middle school mathematics teaching and learning.

BACKGROUND

Recently, I had the opportunity to visit several Japanese schools, both elementary schools (grades 1–6) and lower secondary schools (grades 7–9). A typical mathematics lesson began with the teacher posing a contextualized problem without demonstrating how to solve it, as described in Stigler and Hiebert (1999). The students worked independently on the problem for about ten minutes; the majority of the class period was then spent on discussing, not just sharing, students' solutions. As I observed those mathematics lessons, I noticed students using different diagrams not only to solve the given problems but also to explain their ideas to classmates. Here are some of the ways that students used diagrams.

DOUBLE NUMBER LINE

A double number line is composed of a pair of number lines that are drawn parallel to each other and hinged at 0. Because the scaling on the two number lines is (usually) different, the model can visually show proportional relationships of two quantities. Double number lines are commonly used in Japanese elementary mathematics textbooks (Watanabe, Takahashi, and Yoshida 2010).

In one sixth-grade introductory lesson on "speed," the teacher presented **table 1** and asked, "Which child ran faster?" Students easily decided that Shinya was faster. When the teacher asked them to explain how they knew that Shinya was faster, the class agreed that because both Shinya and Tohru ran the same distance (40 m), Shinya ran in a shorter time period and therefore must be faster.

The teacher then presented the distance and the time for another student, Shigeki, and asked the class to determine which of these two students ran faster, Tohru or Shigeki. (See **table 2**.) Again, the class easily decided that Shigeki ran faster, explaining that since the two students ran for the same amount of time (9 sec.), Shigeki, who ran a longer distance, must be faster.

Then, the teacher posed the main task for the lesson, "Who ran faster, Shinya or Shigeki?" On the basis of the opening discussion of the lesson, the students realized that if they could somehow make either the time or the distance the same, they could decide which student ran faster. One student wanted to make the time the same.

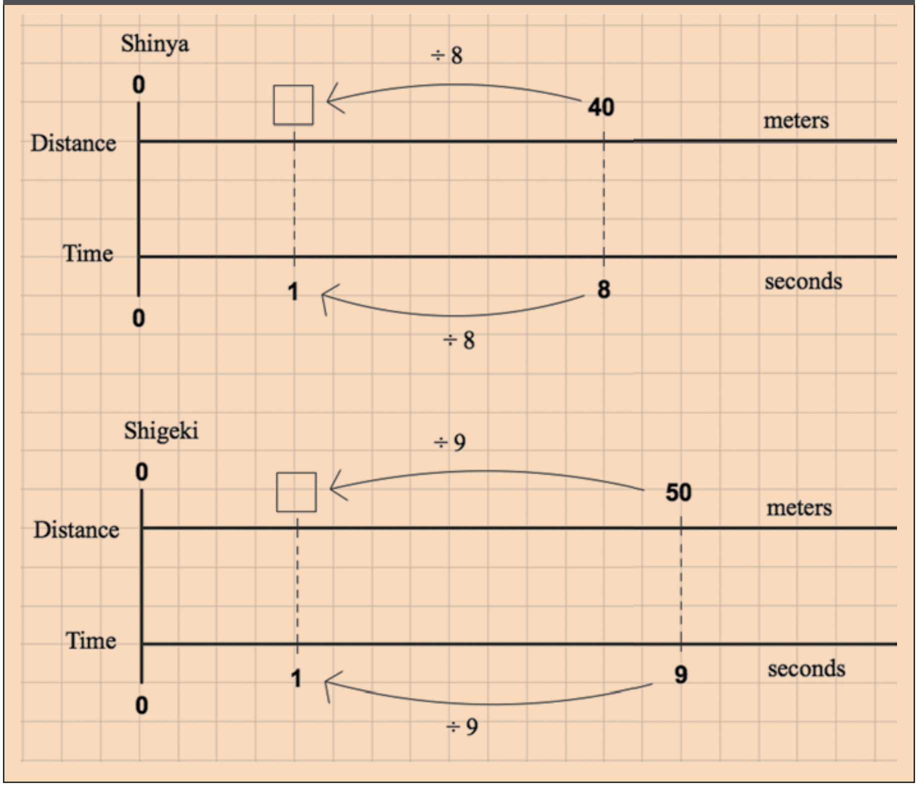
Table 1 Which child ran faster?

	Distance (m)	Time (sec.)
Shinya	40	8
Tohru	40	9

Table 2 Shigeki was determined to be the faster runner.

	Distance (m)	Time (sec.)
Shinya	40	8
Tohru	40	9
Shigeki	50	9

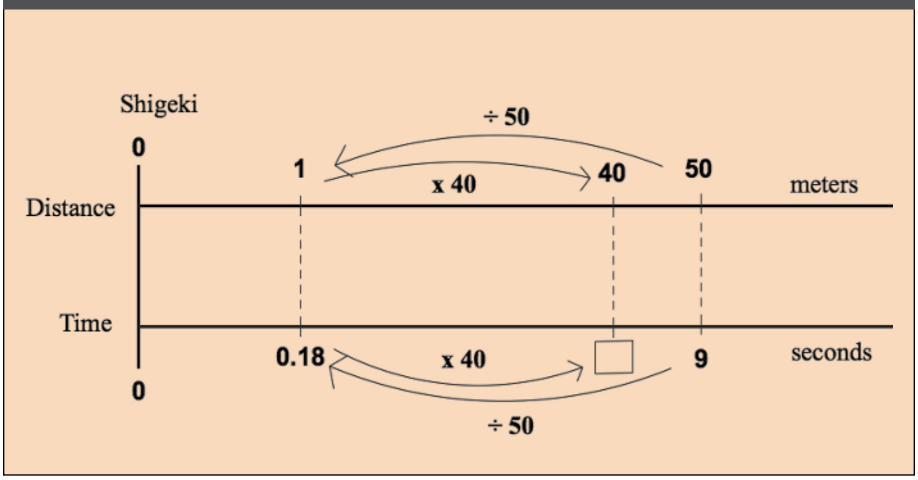
Fig. 1 A sixth-grade student used these diagrams to equate the two students' time.



She decided to figure out how far each student ran in 1 second, so she drew a pair of double-number-line representations, as shown in **figure 1**. She determined that Shinya ran $40 \div 8 = 5$ meters, whereas Shigeki ran $50 \div 9 = 5.6$ meters (rounded to the nearest tenth). Therefore, since the amount of time was the same (i.e., 1 second), Shigeki was faster because he had run a longer distance.

Another student wanted to make the distance the same, that is, to determine how much time Shigeki needed to run 40 meters. He drew the double-number-line representation, shown in **figure 2**, and calculated Shigeki ran for $9 \div 50 \times 40 = 7.2$ sec. Since Shigeki needed less time to run 40 meters than Shinya, the student concluded that Shigeki was faster. Both of these students used double-number-line diagrams as a tool to determine what calculations were needed.

Fig. 2 Another sixth grader used this diagram to determine how much time Shigeki needed to run 40 meters.



TAPE DIAGRAM

A tape diagram is another common visual representation in Japanese elementary mathematics textbooks (Watanabe, Takahashi, and Yoshida 2010) and is similar to the strip (or bar) diagrams in Singaporean textbooks. Although it can be used for a variety of situations, a tape diagram is particularly useful in situations that compare two or more quantities.

In a later lesson on speed in the same sixth-grade classroom, the teacher posted **table 3**. Students were able to determine that the sixth grader was a faster runner because the first grader could only run 80 meters with the time doubled to 16 seconds, whereas the sixth grader could run 100 meters in the same amount of time. The teacher then posed the following problem to the class:

> These two students will be racing on a 120 meter racecourse. They will start at the same time. If we want them to reach the goal at the same time, how far ahead of the sixth grader should the first grader start?

Many students used double number lines to help them figure out how far the first grader could run in the amount of time that the sixth grader could run the 120 meter course. One

Table 3 Who was faster, the first grader or the sixth grader?

	Distance (m)	Time (sec.)
First grader	40	8
Sixth grader	100	16

Fig. 3 A sixth grader used this tape diagram to determine that the sixth grader in the problem must run 30 meters farther than the first grader.

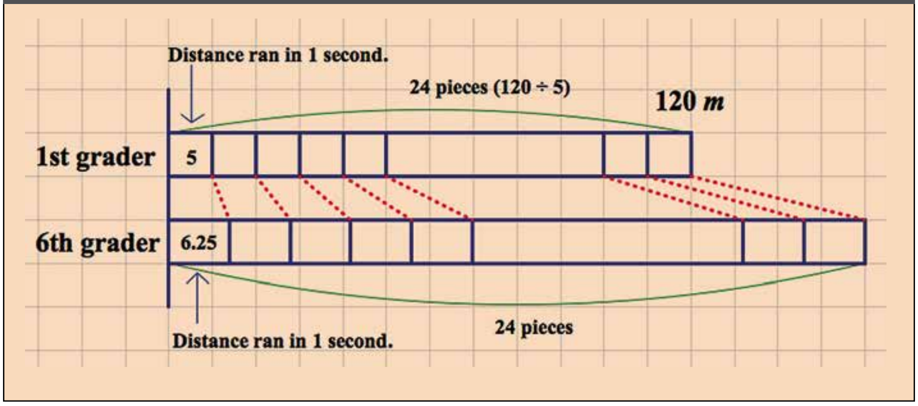
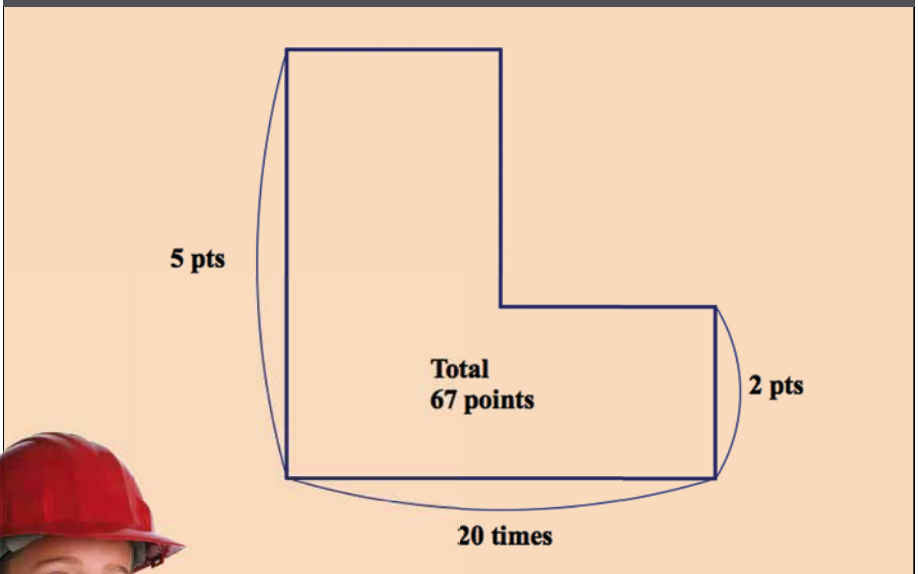
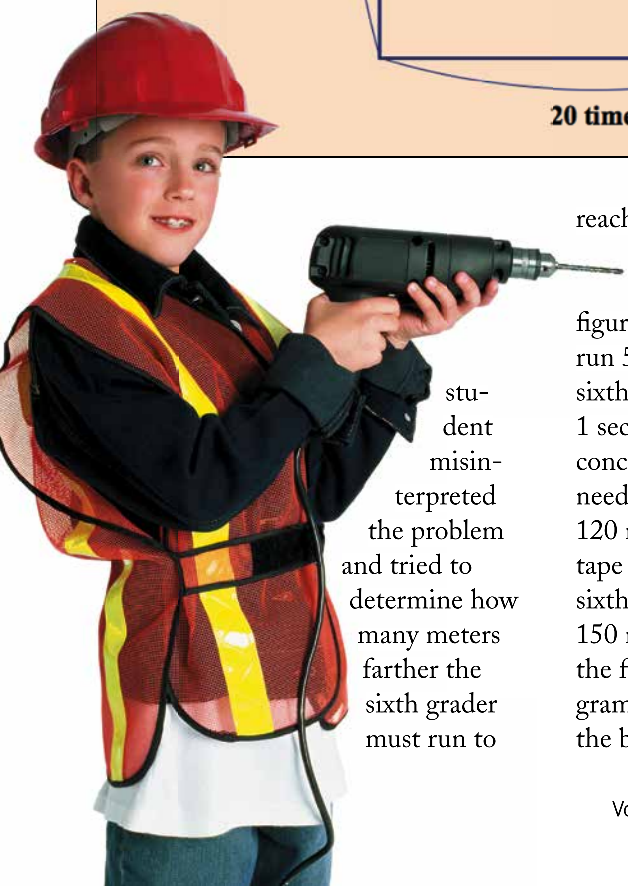


Fig. 4 Some seventh graders used this area model to help them solve the problem about the rock-paper-scissors game.





student misinterpreted the problem and tried to determine how many meters farther the sixth grader must run to reach the goal at the same time the first grader completed the 120 meter course. He first figured out that the first grader could run 5 meters in 1 second, and the sixth grader could run 6.25 meters in 1 second. From that information, he concluded that the first grader would need 24 seconds to complete the 120 meter course. He then drew the tape diagram to figure out that the sixth grader would have to run 150 meters, or 30 meters farther, than the first grader. (**Fig. 3** is the tape diagram model that the teacher drew on the blackboard, based on the student's drawing.) Had he interpreted the problem correctly, this student would have drawn a related yet different diagram. Since the sixth grader would need 19.2 seconds to run 120 meters, he would have drawn 19.2 pieces of blocks for each student. Moreover, the known distance, 120 meters, would be for the sixth grader, not the first grader, as shown in **figure 3**. Then, he would have figured out that the first grader could run 96 meters in that time span (19.2×5). Thus, the first grader must start 24 meters in front of the sixth grader for the two students to reach the finish line at the same time.

AREA MODEL

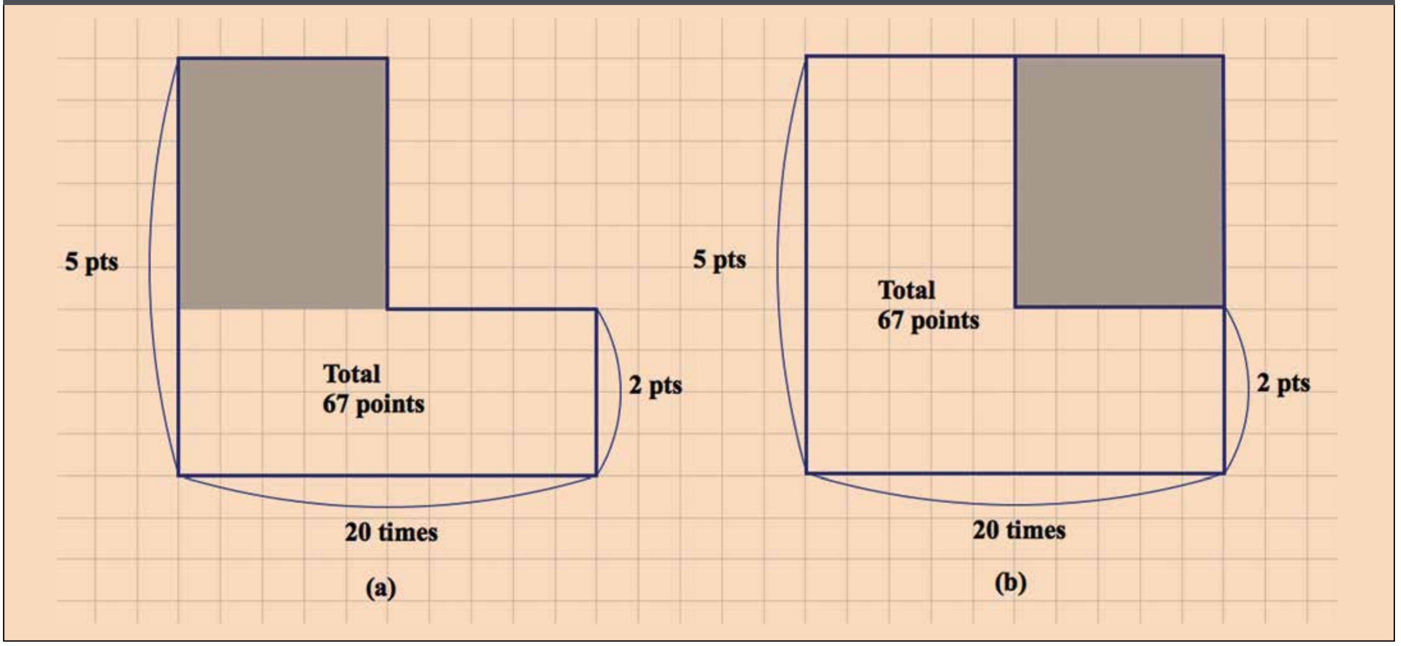
An area model is a useful model for certain multiplicative situations in which one quantity is the product of two other quantities. Readers are probably familiar with using this model to illustrate multiplication of decimal numbers or fractions. However, because it models multiplicative situations, the area model can serve as a powerful reasoning tool for solving certain types of problems involving multiplication and division.

In a seventh-grade lesson on applications of linear equations, the teacher posed the following problem to his students:

> In a rock-paper-scissors game, a winner gets 5 points and a loser gets 2 points. After a man played this game 20 times, his total score was 67 points. How many times did he win, and how many times did he lose?

Because this was only the fourth lesson in the unit of linear equations, many students solved this problem without using an equation. In particular, some students used the area model shown in **figure 4** to help them solve the problem.

Fig. 5 Students solved the rock-paper-scissors game problem in two different ways, both based on the area model.



In this L-shaped diagram, the vertical side on the left represents the points that the player earned by winning a game; the vertical side on the right stands for the points earned by a loss. The vertical side in the middle, therefore, is the difference in points earned between a win and a loss. The horizontal dimension in **figure 4** represents the number of games played: The bottom represents the total number of games played (20); the unlabeled, top-left side corresponds to the number of games won; and the unlabeled, horizontal side to the right of the L-shape stands for the number of games lost. The "area" represents the total points that the person earned.

Using this diagram, some students solved the problem by noticing that the area of the shaded part in **figure 5a** is 27 (i.e., $67 - 20 \times 2 = 27$). Therefore, the number of games that the person won, that is, the dimension of the horizontal side on the top left, is 9 (i.e., $27 \div 3 = 9$). Then, by subtracting 9 from the total number of games played, students found that this person lost 11 games.

Other students determined that the shaded part in **figure 5b** is 33 (i.e., $20 \times 5 - 67 = 33$). Therefore, the number of games that this person lost must be 11 (i.e., $33 \div 3 = 11$). Once again, by subtraction, these students were able to figure out that 9 games were won.

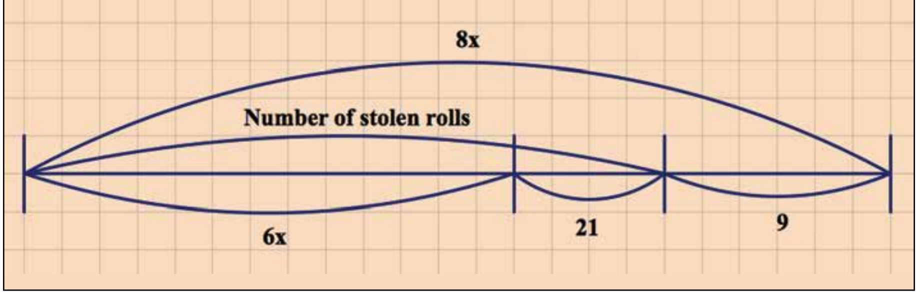
SEGMENT DIAGRAM

Segment diagrams are structurally identical to tape-strip-bar diagrams. Instead of using a thin rectangle to represent a quantity, a segment represents a quantity. Thus, any tape-strip-bar diagram can be replaced with a segment diagram. A different seventh-grade teacher posed the following problem, taken from a Japanese mathematics book published in the seventeenth century:

> One night a group of thieves robbed a clothing store and stole rolls of silk. They hid under a bridge and tried to determine how they should split their loot. One thief said, "If we give 6 rolls to each, there will be 21 extra rolls left, but if we give 8 rolls to everyone, we are 9 rolls short." How many thieves were there, and how many rolls of silk did they steal?

The teacher explained where this problem originated. Although the

Fig. 6 Some seventh graders used a similar diagram to determine the number of thieves and the number of stolen rolls.



Encourage students to include visual representations in their own reasoning toolkits.

context may be socially inappropriate in North America, it is perhaps similar to pirates sharing their loot in Western folktales or fantasies. Teachers may want to alter the problem context if they choose to use this problem in their own classrooms.

Similar to the other seventh-grade class, these students had just recently been introduced to linear equations. Thus, some students used a diagram like the one shown in **figure 6** as they thought about the problem. These students let x be the number of thieves. From this diagram, those students realized that the difference between $8x$ and $6x$ is the sum of 21 and 9. Then, they solved a simple linear equation, $2x = 30$, to find the number of thieves, 15. After that, they calculated the number of stolen rolls to be 111 (i.e., $6 \times 15 + 21 = 111$, or $8 \times 15 - 9 = 111$).

CHALLENGES AND OPPORTUNITIES

The examples discussed illustrate the power of drawings and diagrams as problem-solving strategies and as explanation tools. However, one of the reasons that the students discussed in these examples were able to confidently use these visual representations as tools is because they had been using them since their early elementary school years (Murata 2008; Watanabe, Takahashi, and Yoshida 2010). Thus, one of the challenges for middle-grades mathematics teachers in North America, where these tools are not used as consistently, is how to encourage students to include visual representations in their own reasoning toolkits. One suggestion is to incorporate visual representation tools intentionally when reviewing materials from prior grades. Teachers could give students a hypothetical student's solution to a problem that uses a visual representation tool and ask them to explain how the visual model represents the problem situation. Then, they could follow up by using visual representations to solve other problems.

Because these visual representations are powerful problem-solving tools, they also present a different challenge to mathematics teachers. As discussed, the two seventh-grade classrooms in which students used visual representations were both focusing on applications of linear equations. The teachers hoped that the contextualized problems they posed to the class would help students see the usefulness of linear equations as a problem-solving tool. However, some students already possessed powerful tools and did not need to use linear equations to solve the problems.

Similarly, double-number-line diagrams can be used to represent missing-value proportion problems. Thus, instead of setting up proportions to find the missing value, students can solve the problems arithmetically as prospective elementary school teachers did in Watanabe, Takahashi, and Yoshida (2010). Therefore, the challenge is how to help students who are proficient with these visual reasoning tools learn the power and usefulness of more advanced mathematical tools like linear equations.

As mathematics teachers consider this challenge, keep in mind that these visual representations are adequate when the focus is on solving particular problems. In other words, if the classroom discussion focuses only on the correctness of the answers, these solution strategies would be equally valid. The students would have no motivation to understand or appreciate other strategies, such as linear equations, even if those strategies are more advanced mathematically. On the other hand, class discussions that extend beyond the correctness of answers or build on student solutions—for example, comparing and contrasting a variety of solution strategies or examining mathematical structures of problem situations—may be enriched by these visual representation tools. They may help students

Fig. 7 The teacher anticipated that some students might use the area model to solve the problem.

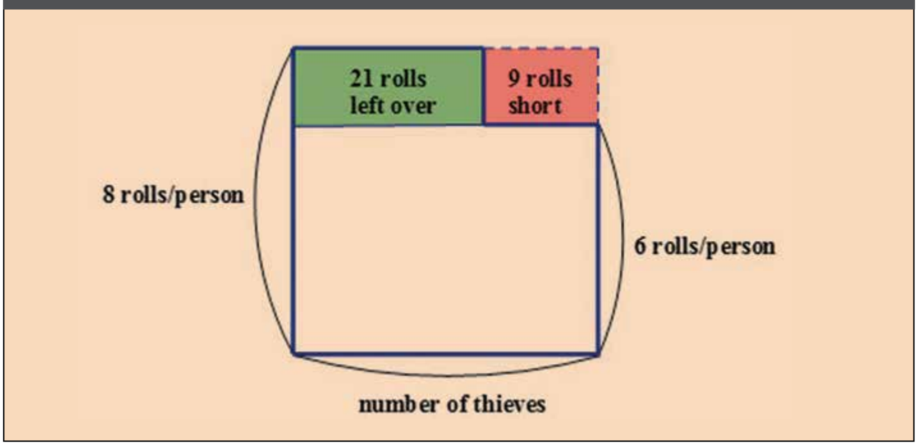
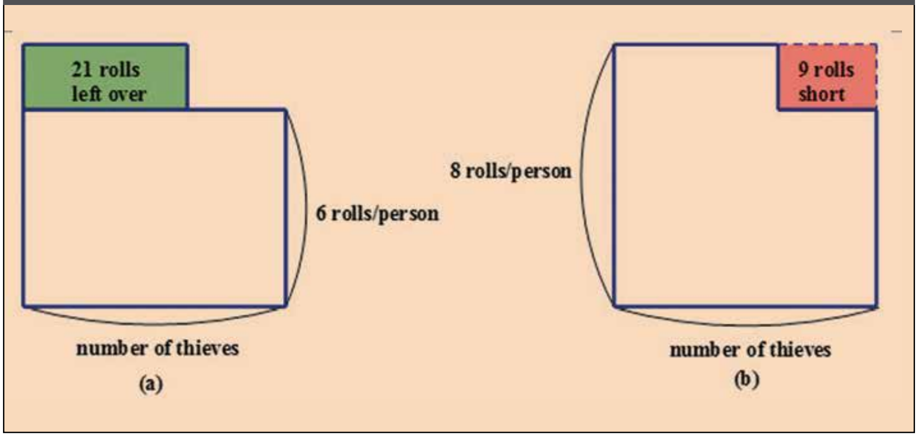


Fig. 8 Area model (a) shows that when 6 rolls were given to each thief, there were 21 rolls left over. Area model (b) shows that giving 8 rolls to each thief meant that 9 more rolls were needed.



make sense of more advanced mathematics. Making connections among various mathematical representations and facilitating meaningful mathematical discourse are two key features of effective teaching of mathematics (NCTM 2014).

These visual representations also offer new opportunities for exploring mathematical relationships. Although none of the students in the second seventh-grade classroom actually used the area model to solve the Silk Thieves problem, the teacher anticipated that some students might use the area model shown in **figure 7**. In this diagram, the vertical dimension represents the number of rolls for each thief, whereas the horizontal dimension represents the number of thieves. The shaded rectangle on the top left represents the extra rolls left over when 6 rolls were given out to all the thieves, and the shaded rectangle on the right represents the 9 roll shortage if 8 rolls were to be given out to the thieves.

Figure 8 shows the two sharing situations separately. The area model shown in **figure 7** can be thought of as the combination of the two models in **figure 8**. From this diagram, the sum of 21 and 9 must equal the product of 2 (the difference in the number of rolls to be given to each thief) and the number of thieves. Therefore, the number of thieves must be 15. Once the total number of thieves is known, we can calculate the total number of stolen rolls to be 111, either by $15 \times 6 + 21$ or $15 \times 8 - 9$.

The fact that this problem can also be represented using the area diagram suggests that the rock-paper-scissors game problem and the Silk Thieves problem have a common mathematical structure. Therefore, teachers could have students explore that mathematical structure. Then, students could be challenged to represent the rock-paper-scissors game problem using the segment diagram. Exploring and using mathematical structures is an important mathematical practice highlighted by CCSSM.

How are these problems related? In the rock-paper-scissors game problem, a common reasoning strategy for solving the problem would be to think about the situation if the man won all his games. In that case, that individual would have earned

The challenge is how to help students who are proficient with these visual reasoning tools learn the power and usefulness of more advanced mathematical tools.

100 points, 33 more points than he actually earned. (See the shaded rectangle in **fig. 5b**.) These are the points that "fell short." However, had he lost all the games, this person would have earned 40 points, when actually 27 more points were earned. These are the "leftover" points. Thus, a parallel to the Silk Thieves problem in the context of rock-paper-scissors would be this:

> A student played the game several times. His total score was 27 more points than what he would have earned had he lost all the games, and it was 33 fewer points than what he would have earned had he won all the games. How many games did the student play, and what was the score?

VISUAL REPRESENTATIONS AS POWERFUL TOOLS

We considered how middle-grades students can use visual representations as powerful problem-solving tools. These examples reaffirm the importance of the systematic and consistent integration of visual representations in mathematics teaching suggested by other scholars. Thus, integrating appropriate visual reasoning tools is one way to make the mathematics teaching principles (NCTM 2014) come to life. Of course, this integration depends on the teachers' ability to use these tools.

If you are not familiar with any of these visual reasoning tools, I encourage you to try them because solving problems will acquaint you with their mechanics. The **sidebar** provides some resources for getting started. A deep examination of these tools will prepare you for helping students see both their power and limitations (NCTM 2014). It will also help students develop the mathematical practice of using appropriate tools strategically (CCSSI 2010).

Getting Started

The following resources are for readers who are interested in learning more about the specific visual representations explored in this article:

- Beckmann (2004) and Ng and Lee (2009) discuss how strip/bar diagrams are being used in Singaporean textbooks.
- Cohen (2013) describes how strip diagrams can be used to help students make sense of proportions. Keep in mind that any tape/strip/bar diagram can be replaced with a segment diagram.
- Beckmann and Fuson (2008) explore how various mathematical representations, including tape diagrams and double number lines, can be useful in middle-grades mathematics classrooms.
- Watanabe, Takahashi, and Yoshida's (2010) offering includes a detailed discussion on double number lines.

Although area models have been used in U.S. mathematics curricula, very few resources discuss how to use them as reasoning tools. A key feature of area diagrams is that they represent three quantities: One of the quantities is the product of the other two. Interested readers may want to keep this feature in mind and try to identify word problems that involve quantities related in this manner.

Note that an English translation of grades 1 through 9 Japanese mathematics textbooks, *Mathematics International* (Tokyo Shoseki 2012), is available in North America.

I hope this article will motivate many teachers to collaborate and devise plans to integrate these tools in such ways that they build student understanding and reasoning (NCTM 2014).

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