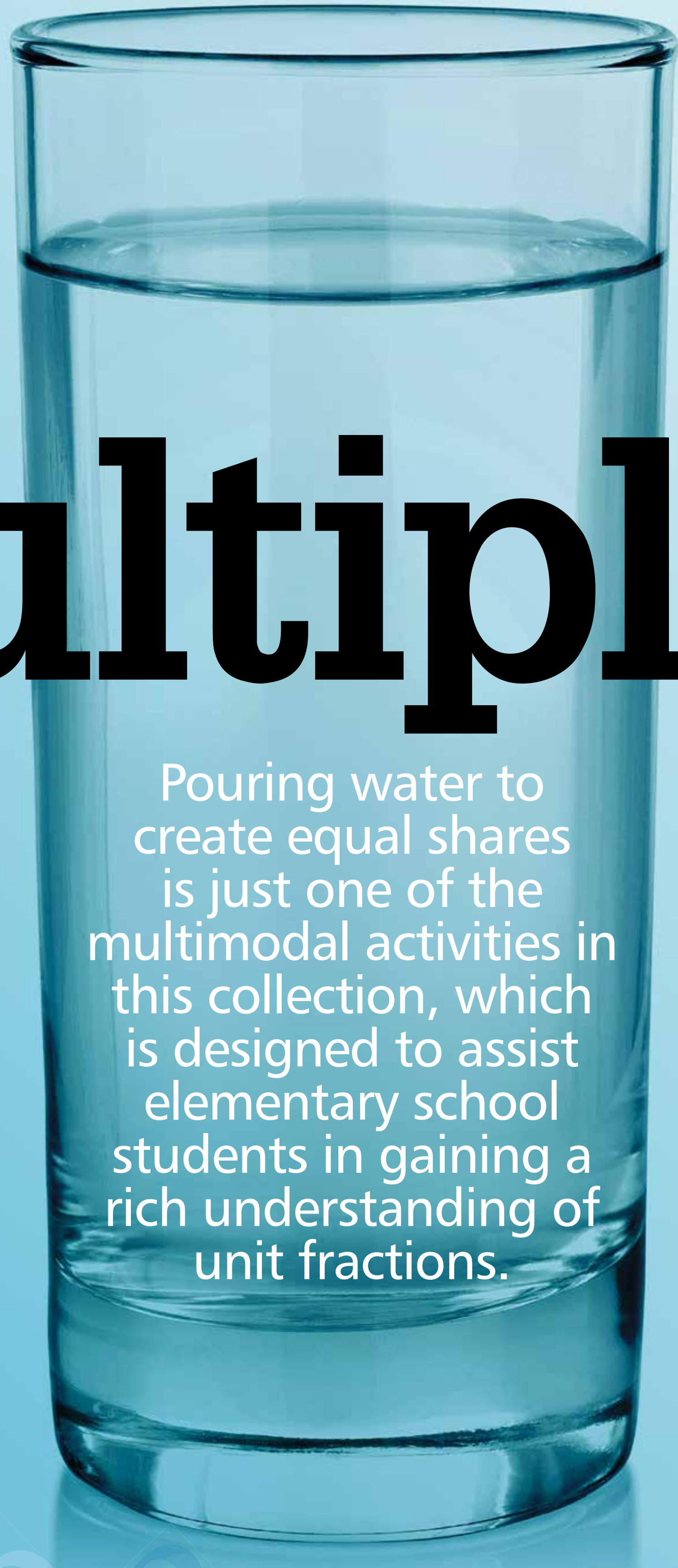
Engaging Students with Multiple



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Pouring water to create equal shares is just one of the multimodal activities in this collection, which is designed to assist elementary school students in gaining a rich understanding of unit fractions.

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Between the Wholes: Promoting Fraction and Decimal Understanding

Models of Fractions

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An understanding of unit fractions, and especially of one-half, one-third, and one-fourth, is crucially important for elementary school children's development of number sense (CCSSI 2010). We describe multimodal activities designed to assist elementary school students in gaining a rich understanding of unit fractions. Research has shown (Zhang 2012) that participation in the activities described in this article helped fifth-grade students develop wider imagery and deepen their understandings with respect to unit fractions concepts. A body of literature is developing that is directed at teachers who wish to fruitfully engage elementary school students in concept-rich fractions activities (e.g., Ball 1993; Ball, Thames, and Phelps 2008; Barnett-Clarke et al. 2010; Chval et al. 2013).

A multiple-embodiment approach to teaching and learning fractions

The concept of a fraction is one of the most important mathematical ideas that elementary and middle school students encounter (Behr et al. 1984). However, the—

> difficulty with the learning of fractions is pervasive and is an obstacle to further progress in mathematics and other domains dependent on mathematics, including algebra. (U.S. DOE 2008, p. 28)

The view that an area-model approach should take priority in thinking about the teaching and learning of fractions has received much attention (Cramer and Henry 2002; NCTM 2000). Students are encouraged to represent fractions with plastic fraction pieces in "fraction kits" and draw or shade circular or polygonal (especially square and rectangular) representations of fractions.

Research indicates (Clements and Lean 1994; Samsiah 2002; Zhang 2012) that an overreliance on area-related contexts for teaching fraction concepts can result in students not coping well with fraction tasks that are presented in non-area situations. Samsiah (2002) reported that the students in her research could answer questions based on the area model but could not transfer that knowledge to related, real-life situations. Zhang (2012) investigated the knowledge of fractions of fifth graders in the United States who had been taught the concept by methods in which area models had been almost exclusively emphasized and found that although the students could cope well with area-model tasks, many of them struggled when dealing with corresponding fraction tasks presented in non-area-model contexts.

FIGURE 1

Develop a number line using paper strips to emphasize to students that the whole is the length between 0 and 1 on any of the three paper strips.

Some researchers have pointed to the importance of a multiple-embodiment approach to teaching and learning fractions so to assist students in deepening conceptual understandings of fractions (Moss and Case 1999; Zhang 2012). Moss and Case (1999) attributed the difficulties that many children experience when attempting to learn fractions to an overemphasis on area models; they advocated the use of "some alternative form of representation" (p. 144). Clarke, Roche, and Mitchell (2008) expressed regret that many students regard the area model as the only model of a fraction and proposed that if students are to understand fractions conceptually, they must become acquainted with a wider range of models.

From a mathematical perspective, the Common Core State Standards for Mathematics (CCSSM) has emphasized the importance of presenting fractions first and foremost as numbers with unique places on what will ultimately become known as "the real number line" (CCSSI 2010; Wu 2014). In the classroom, implementations of each of the six activities described below, the idea of placing fractions on a number line, was consciously and strongly emphasized. In any activity, identifying the meaning of the symbol m/n is important, where m and n are natural numbers and the result of a whole (or unit) having been partitioned into n equal parts, and m of those parts are being considered. The part-whole aspect of the concept definition is fundamental and must be emphasized in any activity aimed at developing sound fraction concepts.

Experiencing different representations of the same concept can help students abstract structural similarities and develop conceptual understandings for fractions. Students can be given opportunities to use different models when learning fractions. In what follows, we describe six multimodal activities that demonstrate how to teach fractions like one-half, one-third, and one-fourth.

Three linear-model embodiments

Activity 1: Developing a number line using paper strips

Three narrow paper strips, each having the same length but different colors—let's say green, yellow, and purple—and a sheet of paper are prepared and made available to pairs of students. On the sheet of paper, a line segment, which is the same length as the three colored strips, is drawn, with the symbol "0" at one end and "1" at the other end (see **fig. 1**). This "zero-to-one" line segment is used to emphasize that the unit involved, the "whole," is the length between the positions for 0 and 1 on any one of the three paper strips.

Working in pairs, students *fold* the green paper strip into two equal parts, the yellow strip into three equal parts, and the purple strip into four equal parts. Then they unfold the pieces of paper and use them to locate positions for 1/2, 2/2, 1/3, 2/3, 3/3, 1/4, 2/4, 3/4, and 4/4 on the number line drawn on the sheet of paper. Before students attempt to locate the positions for the fractions on the number line, the teacher emphasizes that 2/3, for example, means "two lots of 1/3."

Activity 2: Folding a piece of rope

Working in pairs and using a piece of rope (about one yard long) and three different colored markers (red, blue, and green), students fold and then indicate the following: (a) exactly one-half of the piece of rope with a red mark on the rope; (b) exactly one-third and two-thirds of the piece of rope with blue marks; and (c) exactly one-fourth, two-fourths, and three-fourths of the piece of rope with green marks. Each pair of students then takes its piece of rope and checks whether the red, blue, and green marks correspond to the correct positions of 1/2, 1/3, 2/3, 1/4, 2/4, and 3/4 that have been placed on a number line drawn on the board (which has the symbol "0" at one end and "1" at the other). The distance from "0" to "1" is one yard; this constitutes the "whole." As with activity 1, the meaning of a fraction like 3/4 as "three lots of 1/4," is emphasized. Activity 2 is structurally and materially like activity 1 deliberately—both are likely to guide students toward the realization that a fraction is a number with a unique position on a number line. After students become familiar with activity 2, they can be invited to work on two in-class tasks (see **fig. 2**).

Activity 3: Developing a human number line

Thirteen students are called to the front of the classroom and are asked to space themselves evenly across the room, facing the class. The student on the left should hold a card showing "0" and should remain standing throughout the activity. The student on the right should hold a card showing "1." The other students in the class are invited to be "expert observers." The teacher draws attention to the fact that twelve spaces are between zero and one; through questioning, the class concludes that each space has a length of one-twelfth of the whole. Through further questioning, the class determines that successive students should be called *one-twelfth*, *two-twelfths*, *three-twelfths*, and so on.

The activity continues with the twelve active participants (but not "0") being asked to crouch down. Emphasize that the "whole" is the distance between the students representing zero

FIGURE 2

The second activity, two fraction tasks using a piece of rope, is intentionally structured like the first activity to guide students toward understanding that a fraction has a unique position on a number line.

1. Find ***two*** ways to tell your partner, in words, how you could use the number line on the board and your piece of rope to answer the following:

   $2 \times \frac{1}{2} =$

   $3 \times \frac{1}{3} =$

2. Draw pictures and explain how you folded the rope for each of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

and one. Then call out fraction names: one-half, one-twelfth, one-third, two-thirds, three-thirds, one-fourth, two-fourths, three-fourths four-fourths, five-twelfths, and so on. As each fraction name is called out, the "right" number of students—always starting at the zero end of the twelve students—stand up to represent the fraction. The response of standing, or not standing, helps students recognize that five-twelfths, for example, means that the distance between "0" and "5/12" is the sum of five separate distances of one-twelfth. It is also the total distance between "0" and "5/12." Students can be invited to reflect on what "0/12" might mean.

This human number-line activity has the potential to facilitate mathematically rich thinking among young learners. Ideas of equivalent fractions can be discussed ("What is another name for nine-twelfths?"), and after a short time, the "expert observers" should be invited to suggest fraction names. The task can be varied so that, for example, nine students are at the front of the class (instead of thirteen students). In our experience, the activity assists students with recognizing that once the positions for zero and one are fixed, fractions have unique positions on a number line.

After completing activities 1, 2, and 3, ask students to discuss review questions or extensions (see **fig. 3**) so that they will practice, consolidate, and reflect on how to extend their knowledge and skills with respect to the number line.

A perimeter-model embodiment

Activity 4: Playing "fractions baseball"

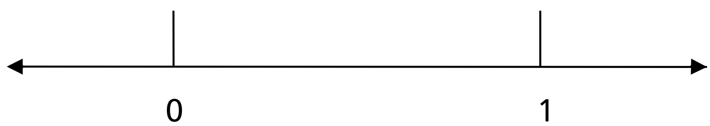
Four students are invited to make the shape of a square, remembering to create four equal sides using a piece of rope about eight yards long. After doing this, they fix the square to the floor. Then an "imaginary game of baseball" can be played, beginning with two volunteers who are chosen to act as "pitcher" and "batter." The pitcher stands in the middle of the square and pretends to pitch to the batter, who stands at one corner (which corresponds to "0" and also to "1") and swings, pretending to try to hit the ball. Then the batter runs around the sides of the square a distance that should correspond to a fraction name (e.g., "one-fourth") called out by the teacher. Thus, if "one-fourth" were called, then the batter would run, in a nominated direction (usually counterclockwise, to correspond to the game of baseball) along one side of the square.

Then the next batter would take the "plate," and the exercise would be repeated, but three students would be involved (the pitcher and the first and second batters). During the game, calls of "one-fourth," "two-fourths," "three-fourths,"

FIGURE 3

After students complete the first three activities, ask them to practice using their knowledge and skills with respect to fractions on a number line.

The number line below has space for some numbers less than 0 and some numbers more than 1. On this number line, show where $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ should be placed. Where should $\frac{2}{3}$ and $\frac{3}{4}$ be placed? Where should $\frac{1}{6}$ and $\frac{1}{8}$ be placed? Where should the fraction represented by $\frac{9}{8}$ be placed? Where should the fraction represented by $\frac{24}{12}$ be placed?



Students have a chance to experience representations of fractions, such as 1/4, 2/4, 3/4, and 4/4, as they walk the bases of a mock baseball game.

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Pouring water to create equal shares precedes writing fraction stories about sharing lemonade.

and "four-fourths" can be made, as well as "foul" and "strike one," and so on. Students can also be asked to reflect on what the calls "one-half" and "one-third" might mean in the context of the game.

With activity 4, emphasize that the "whole" is the *total* distance around the square (or "diamond"). The square is like a number line depicting numbers from 0 to 1; the number line has been folded so that it has four equal sides. Also, three-fourths, for example, corresponds to running three of the four sides. Students can be invited to reflect on what one-half means and on why one-half is equal to two-fourths.

As an extension to the activity, use the same rope to create a boundary in the shape of an equilateral triangle and fix it on the floor. Students could play a similar game of Fractions Baseball. Present activities like those with the square (or diamond) baseball game and invite students to "run" one-third, two-thirds, or three-thirds of the way around the triangle. Ask students to reflect on what one-half, one-fourth, and one-sixth might indicate in this activity. Moreover, have them explain how they link called-out fraction names with the sides of the triangle.

To make the activity more interesting, form two teams of students, and develop a procedure for drawing "calls" that are written on cards and placed in a bag. Have teams keep score. Elementary school students enjoy this game and can learn a lot about fractions by playing it.

A capacity-model embodiment

Activity 5: Pouring water

For this activity, students work together in groups of three. Each group is allocated a full jug of water and four clear glasses (each glass having the same height and radius and having vertical rather than oblique or curved sides). Students pour a full glass of water and then, by pouring, share it as equally as they can among the three other glasses. They then pour the water back together and develop four stories involving pouring lemonade, individually or with their group, that correspond to the following:

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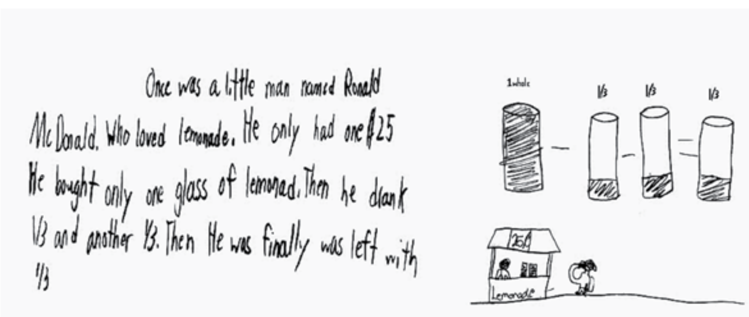
The Pouring Water task has the added benefit of developing students' precision in problem solving.

FIGURE 4

Three fifth-grade students created a group story illustrating 1 – 1/3 –1/3.

Work in groups of three. Talk with the others in your group, and work out the value of $1-\frac{1}{3}-\frac{1}{3}$ by pouring water. Then make up a "pouring lemonade" story and draw a picture about $1-\frac{1}{3}-\frac{1}{3}$.



(a) 1/3 + 1/3 + 1/3 = ___
(b) 1/3 + 1/3 = ___
(c) 1 – 1/3 = ____
(d) 1 – 1/3 – 1/3 = ____ (see **fig. 4**)

Students conceived of the "whole" as one full glass of lemonade and linked this to the idea that a glass of lemonade would cost 25 cents. You might ask students to also work on other fraction tasks; for example, encourage them to make up, write, and illustrate a "sharing lemonade" story that demonstrates how one-third is more than one-fourth.

A discrete-model embodiment

Activity 6: Sharing precious diamonds

Students form pairs, and each pair receives twelve discrete identical blocks (described as *precious diamonds*). Emphasize that although there are twelve individual blocks, in fact, the “whole” comprises all twelve blocks. Pairs are asked to show and explain how the twelve diamonds can be shared equally among two, three, and then four friends. Experience has shown that the most common strategy used is to “deal” blocks, successively, to individual “friends;” but teachers should watch pairs to see which of them, if any, form and then partition rectangular arrays. If such partitioning does not occur intuitively, then suggest to pairs of students that they think about how such partitioning could be associated with fractions like one-half, one-third, one-fourth, and so on. Afterward, students can solve the two tasks (see **fig. 5**) individually and discuss with their partner how they determined their answers.

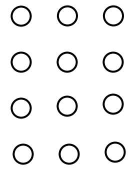
Reviewing

After the sequence of activities has been completed, distribute a review sheet to students (see **fig. 6**). It might also be set up as a 6 × 6 grid, with the symbols 1/2, 1/3, 1/4, 2/3, 3/4, and 1 on the left side of the sheet and, to the right of these symbols, five pictorial representations featuring distances on a number line, around a square, around an equilateral triangle, the amount of fluid in a glass, and twelve identical objects. Ask students to indicate how one-fourth could be

FIGURE 5

Before individuals solve the discrete model tasks, have student pairs partition and discuss rectangular arrays.

1. Suppose you arranged the 12 precious diamonds like this:



Draw a frame around the number of diamonds that would show $\frac{1}{2}$ of 12. Also, draw frames around the number of diamonds to show $\frac{1}{3}$ of 12 or $\frac{1}{4}$ of 12.

2. Draw arrays to show how to find the values of $\frac{1}{3}$ of 6, $\frac{1}{5}$ of 15, and $\frac{3}{4}$ of 20.

FIGURE 6

Below is a representation of one row of the review sheet used by the authors, five contexts to help students develop multiple representations of the fraction one-fourth.

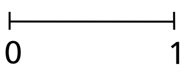
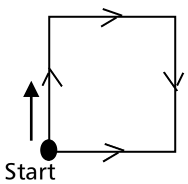
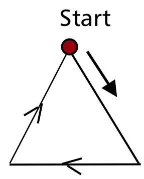
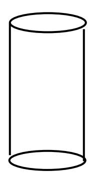
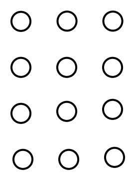
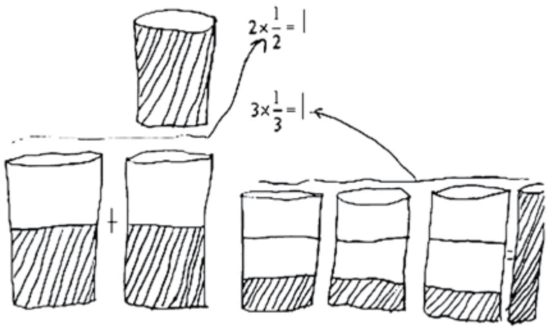
Symbol	Fraction on a number line (linear model)	Fraction of the path around a square (perimeter model)	Fraction of the path around an equilateral triangle (perimeter model)	Fraction of a glass of water (capacity model)	Fraction based on 12 discrete objects (discrete model)
$\frac{1}{4}$					

FIGURE 7

This student could have improved her illustration if she had shown all the glasses as having the same height and volume.

(a) She drew two half-full cups with one full cup to illustrate 2 × 1/2 = 1. Three cups, each filled with one-third cup of liquid, and one full cup illustrate 3 × 1/3 = 1.



(b) She used the perimeters of an equilateral triangle and a square to find 1/3 and 1/4 of a piece of rope and showed how she had folded the triangle and square.

represented with the different models—linear, perimeter, capacity, and discrete models. As students complete the sheet and explain their solutions, select students to fill out the same table drawn on the board.

General comments on the pedagogical approach

When a new activity is introduced to students, it is important to invite them to make connections between the new activity and those activities dealt with before, and to reflect on whether the various activities have something in common. Such an exercise can assist students in abstracting the symbols of fractions like one-half, one-third, and one-fourth out of the specific contexts and to build up a structural synthesis of conceptions of the associated fractions, thereby bolstering their understandings of fraction concepts.

Fifth graders who had engaged in these activities applied what they had learned in the Pouring Water and Playing Fractions Baseball activities to answer the requests in **figure 2**, which were used in the activity of folding a piece of rope. To illustrate 2 × 1/2 = 1, a student drew two half-full cups with one full cup. To illustrate 3 × 1/3 = 1, she drew three cups, each filled with one-third cup of liquid, and one full cup (see **fig. 7a**). She could have improved her illustration if she had shown all the glasses as having the same height and volume. She used perimeters of an equilateral triangle and a square to find one-third and one-fourth of a piece of rope and drew details of how the triangle and square had been folded (see **fig. 7b**).

Promoting high-quality learning

An overemphasis on the area-model approach to teaching and learning fractions can hinder students' conceptual development of fractions. Research (e.g., Zhang 2012) suggests that a multiple-embodiment approach is more likely to promote high-quality student learning. The variety of models included in the six activities in this article can assist students in developing comprehensive concept images of fractions and facilitate their conceptual understandings of fractions. In Zhang's (2012) study, which featured a treatment and a control group from two fifth-grade classes at the same school, students who participated in the activities developed much deeper conceptual understandings—as well as richer and coordinated concept images of one-fourth, one-third, one-half, two-thirds, and three-fourths—than students in the control group who had not participated in the activities. That said, the teachers who use these activities must emphasize what the "whole" is in each of the activities and how the same conceptual ideas are involved even though different embodiments of the concepts are presented.

Common Core Connections

3.NF.1 4.NF.1
3.NF.3 4.NF.2
3.NF.2

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