The Rich

Fifth graders explore approaches to solving a division-of-fractions problem introduced within the context of hot chocolate servings.

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ness of Children's Fraction Strategies

Laura B. Kent, Susan B. Empson, and Lynne Nielsen

How would you decide when to give your students a problem that involves division of fractions, such as this Hot Chocolate problem:

> You have 4 2/3 cups of hot chocolate powder. Each serving requires 2/3 cup hot chocolate powder. How many servings can you make?

What kinds of strategies do you anticipate they would use, and what do you think they could learn?

In this article, we discuss a special type of multiplication-and-division-of-fractions problem that elementary school teachers can use to promote children's understanding of fractional quantities and their relationships. These problems are accessible to students working at different levels of fraction understanding, and they can be solved without the use of standard algorithms for multiplying and dividing fractions. Encouraging children to model the quantities and relationships in the problem situation helps them build a strong foundation for understanding fractions (Empson and Levi 2011; Warrington 1997).

This special type of problem includes equal-groups situations with a whole number of groups and a fractional amount in each group, such as the Hot Chocolate problem (Empson and Levi 2011). We call these *multiple groups problems*. In this article, we discuss how multiple groups problems can be used to promote the development of children's understanding of fractional quantities and their relationships before the introduction of generalized procedures for multiplying and dividing fractions. This type of story problem is appropriate for students of any grade level who are able to model fractional quantities and link these models to a problem situation.

Students' strategies for the Hot Chocolate problem

Children in the elementary grades can solve fraction story problems by drawing on their informal understanding of partitioned quantities and whole-number operations (Empson and Levi 2011; Mack 2001). Given the opportunity, children use this understanding to model fractional quantities, such as 1/4 of a quesadilla, and reason about relationships between these quantities, such as how much quesadilla there would be if 1/4 of a quesadilla, 1/4 of a quesadilla, and 1/4 of a quesadilla were combined.

To explore these ideas, we return to the Hot Chocolate problem and share four strategies generated by students in a fifth-grade class. When the classroom teacher, Mrs. Gardner, posed this problem, she deliberately did not show a strategy because she wanted to give students the opportunity to make sense of the problem on the basis of their own understanding of fractions. To engage her students, she encouraged them to relate to the problem situation by imagining a time when they made hot chocolate using a mix. Students shared their experiences, and Gardner helped them visualize a container of powder and the process of making a serving. Her goals were to support her students in making sense of the problem and generating a strategy that they could explain and justify.

FIGURE 1

Maggi used a direct-modeling strategy for solving the Hot Chocolate problem.

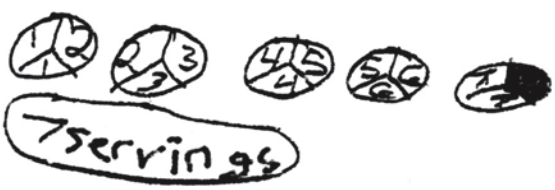
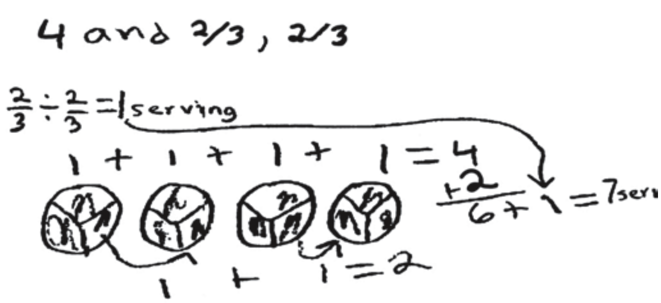


FIGURE 2

Gabriela also used a direct-modeling strategy to solve the Hot Chocolate problem, but she counted servings differently.



Direct modeling of fractional quantities

Gardner's fifth graders used a variety of strategies that revealed a range of understanding of fractional quantities and their relationships. We present four strategies to illustrate this range, which is typical of what teachers can expect third, fourth, and fifth graders to do for problems like this one (Empson and Levi 2011).

Maggi solved the Hot Chocolate problem by directly modeling the quantities in the situation (see **fig. 1**). *Direct modeling* describes student-generated strategies that involve representing all the quantities in a way that reflects the structure of the problem situation (Carpenter et al. 2015). Maggi represented each cup with a circle. She then partitioned each cup into thirds and shaded the last third in the fifth cup to show that it was it was not part of the 4 2/3 cups. To determine how many servings could be made by 4 2/3 cups of powder, Maggi numbered the thirds to group them into servings of 2/3 cup each. Thus, "1, 1" in her drawing represents the first serving, "2, 2" the second, and so on, up to "7, 7" for the seventh serving.

Maggi's strategy reveals her understanding of each cup as a whole unit and her implicit understanding that a serving of 2/3 cup can be decomposed into 1/3 cup and 1/3 cup. This understanding and her representation of the situation helped her determine that there would be seven servings of hot chocolate.

Gabriela also solved the Hot Chocolate problem by directly modeling the quantities (see **fig. 2**). Additionally, she recognized right away that 2/3 cup from the quantity 4 2/3 cups would make one serving and wrote the number sentence 2/3 ÷ 2/3 = 1 to show her thinking. She then drew the remaining 4 cups of chocolate powder similarly to Maggi but counted the servings differently. First, she counted one serving from each of the 4 cups, as shown by her shading of 2/3 of each cup and the equation 1 + 1 + 1 + 1 = 4 above the cups. She then counted the final servings by combining 1/3 and 1/3 from each of 2 cups, twice, as shown by 1 + 1 = 2 below the cups. She added her counts to find a total of seven servings.

These direct-modeling strategies reflect children's early understanding of fractional quantities as a part of a whole and their understanding of division in terms of counting these quantities. This early understanding encompasses relationships that form a necessary foundation for more sophisticated understanding of fractions. Specifically, both Maggi and Gabriela had to understand and be able to reason about the quantity 2/3 cup as a combination of unit-fraction quantities, 1/3 cup and 1/3 cup. They also had to be able to differentiate between what counted as a cup and what counted as a serving within these drawn quantities. Further, identifying a serving required some flexibility, in that a serving of 2/3 cup could be taken entirely from a single whole, or it could be composed of 1/3 cup and 1/3 cup from two different wholes of the same size. Engaging in direct-modeling strategies presents opportunities for students like Maggi and Gabriela to build their understanding of fractions as quantities and of the relationships between unit fractions and other fractions (Empson and Levi 2011).

FIGURE 3

Jake's strategy was repeated addition.

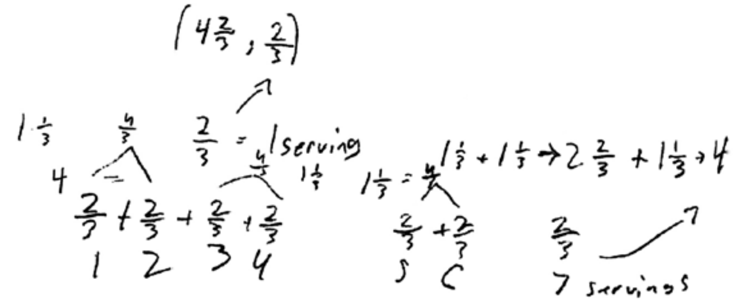
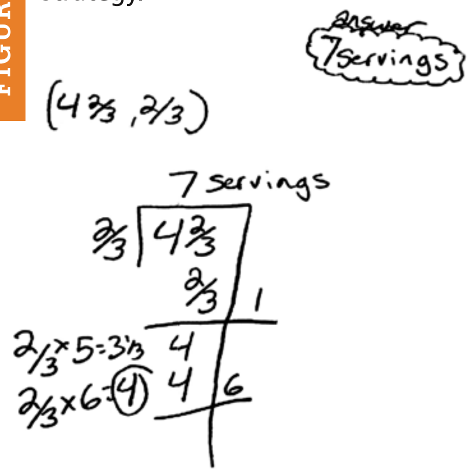


FIGURE 4

Olivia used the partial-quotients strategy.



Using fraction relationships

Other students in Gardner's class approached the Hot Chocolate problem by relating fraction division to repeated addition or multiplication, using a more abstract understanding of fractions. These students did not need to represent the part and the whole as drawn quantities to reason about fraction relationships.

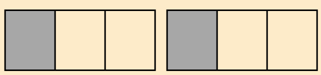
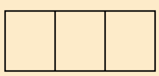
Jake solved the problem by using repeated addition (see **fig. 3**). He represented and reasoned about the fractional quantities symbolically. To successfully use addition to solve the problem, he had to keep track of when to stop adding groups of 2/3 and how many groups of 2/3 he had added. He started by adding four groups of 2/3, using a doubling strategy: 2/3 + 2/3 + 2/3 + 2/3 = 1 1/3 + 1 1/3. His addition reveals his understanding of relationships between fractions and whole numbers. For example, he added 2/3 and 2/3 and got 4/3, which he expressed as 1 1/3, because he knew that a whole cup consists of three groups of 1/3 cup. Using mixed numbers in this way helped him find the sum by adding whole numbers and fractions separately: 1 1/3 + 1 1/3 = 2 2/3. He saw at this point that he could add another grouping of 1 1/3 to get to 4 cups of chocolate powder: 2 2/3 + 1 1/3 = 4. Finally, he indicated that the remaining 2/3 cup of chocolate powder could make one more serving. Jake used addition to solve the problem, but he did not use a standard algorithm for adding fractions. Instead, he used what he knew about fractional quantities and their relationships to produce a mixed-number amount. This strategy provided an opportunity for Jake to work on combining fractions and to reason about relationships between fractions and mixed numbers. When he shared his strategy with the rest of the class, he showed how he numbered each group of 2/3 to keep track of how many servings there were as he added the fractional amounts.

Olivia related the division to multiplication (see **fig. 4**), by estimating how many groups of 2/3 cup would make 4 2/3 cups. Like many students in Gardner's class, she thought about 4 2/3 cups as 4 + 2/3 and realized that one serving would be possible with the 2/3 cup. Because she did not know how many servings of chocolate powder were needed for the remaining 4 cups, she estimated. She reasoned that she would be able to make at least five servings, because each cup could make at least one serving, with some thirds left. She multiplied 2/3 by 5 to get 3 1/3. She explained, “Five groups of 2/3 is the same as ten groups of 1/3. I know 1/3 × 9 is 3, and then 1/3 more would be 3 1/3.” When she saw that this amount was 2/3 away from 4, she realized that she needed six groups of 2/3, rather than five groups, and that 2/3 × 6 = 4. Olivia’s strategy reveals an understanding of the inverse relationship between multiplication and division. Rather than compute 4 2/3 ÷ 2/3 = *c*, she solved the problem by thinking of 2/3 × *c* = 4 2/3. She used a partial-quotients notation to organize her approach to solving the problem. Although this notation is usually reserved for multidigit, whole-number division problems, Olivia used it in a novel way to notate her strategy with fractions, suggesting that she understands fractions as quantities on which she can operate just as she operates on whole numbers.

If students do not see fractions as quantities, they have difficulty making sense of operations on quantities, such as adding or multiplying.

TABLE 1

Students’ strategies for the Hot Chocolate problem reflect their developing understanding of fractions as quantities and their use of fundamental fraction relationships.

Fractional quantities and their relationships	Possible representation
Model a fraction, such as 2/3 cup, as a drawn quantity.	
Two-thirds cup can be decomposed into one-third cup and one-third cup.	2/3 = 1/3 + 1/3
One cup is equal to three groups of one-third cup.	 1/3 1/3 1/3 1 = 3 × 1/3
A mixed number can be decomposed into a whole number and a fraction.	4 2/3 = 4 + 2/3
Multiplication is the inverse of division.	4 ÷ 2/3 = *c* 2/3 × *c* = 4

Reflecting on students’ strategies

The strategies that students used allowed Gardner to address a number of mathematical content and practice goals. After each student had a chance to solve the problem individually, Gardner organized a discussion of selected strategies to allow students to reflect on one anothers’ strategies and extend their understanding (Kazemi and Hintz 2014).

For example, as Maggi shared her direct-modeling strategy and Jake shared his repeated-addition strategy, Gardner asked the class to find Jake’s sum of 2/3 + 2/3 in Maggi’s strategy. She also asked how Maggi’s representation of seven groups could be found in Jake’s strategy. This discussion helped Maggi see how the quantities she had directly modeled could be symbolized, and it helped Jake communicate how his addition strategy worked. When Olivia shared her partial-quotients strategy, the class discussed how operations on fractions and whole numbers are connected. **Table 1** lists some of the fraction content evidenced in the children’s strategies for the Hot Chocolate problem. This content reflects the children’s developing understanding of fractions as quantities and their use of fundamental fraction relationships.

Further, as students engaged in solving and discussing the hot chocolate problem, they used a number of the Common Core’s Standards for Mathematical Practice (SMP) (CCSSI 2010, pp. 6–7). For example, because Gardner did not demonstrate a strategy to use to solve the problem, students had to make sense of the problem and persevere in solving it (SMP 1). She chose a context—mixing hot chocolate—that elicited students’ quantitative reasoning. For some children, such as Maggi and Gabriela, this reasoning was connected to models. For others, such as Olivia and Jake, it was abstract. Olivia’s and Jake’s strategies also made use of structural aspects

TABLE 2

The sample multiple groups problems below are ordered according to their increasing difficulty.

Measurement division	Multiplication
Mom has 10 grilled cheese sandwiches. If she wants to give each child at the party a serving of 1/2 sandwich, how many children can have a serving of grilled cheese sandwich?	Dad wants to serve each child at the party 1/4 sub sandwich. If 8 children are coming to the party, how many sub sandwiches should he make?
It takes 3/4 cup sugar to make a batch of cookies. If the baker has 15 cups of sugar, how many batches of cookies can he make?	The baker wants to make 9 batches of cookies. Each batch uses 2/3 cup sugar. How much sugar does the baker need to make all the batches?
A carpenter is making dog houses that each requires 4/5 pint of paint. If he has 10 2/5 pints of paint altogether, how many dog houses can he paint?	A carpenter wants to make 12 dog houses. If each dog house requires 2/5 pint of paint, how much paint will the carpenter need?

of fraction operations, such as when Olivia related 5 × 2/3 to 10 × 1/3 and when Jake added mixed numbers by combining whole numbers first, then fractions. Engaging in practices such as these supported students in exploring and reflecting on their understanding of fraction quantities and their relationships.

Promoting rich understanding of fractions

One of the most persistent difficulties with fractions is that students do not have enough of the kinds of experiences they need to make sense of fractions as quantities (Empson and Levi 2011; Hackenberg 2013). If students do not see fractions as quantities, they have difficulty making sense of operations on quantities, such as adding or multiplying.

Teachers can use multiple groups problems, such as the Hot Chocolate problem, to build children's understanding of fractions as quantities. Children do not have to be able to multiply or divide fractions to engage productively in solving multiple groups problems, as Maggi's, Gabriela's, and Jake's strategies, in particular, show. As long as children can create models of fractional quantities and explain how they relate to a situation, they are likely to be able to generate strategies for multiple groups problems.

Multiple groups problem types include measurement division and multiplication story problems (see **table 2**). In both cases, the problem situation has a whole number of groups, with a fractional quantity in each group. (The total amount in all the groups may be a whole number or a mixed number.) In a measurement division problem, the number of groups is unknown. In a multiplication problem, the total amount is unknown. The relative difficulty of a multiple groups problem can be adjusted by changing the number selection. For example, children tend to find it easier at first to solve problems that involve a unit-fraction quantity, such as 1/4 sub sandwich, as the amount in a group, and harder to solve problems that involve less familiar fractional quantities, such as 2/5 gallon of paint.

The strategies used by Gardner's students illustrate the richness of children's thinking about fractional quantities and their relationships. All students have the potential to generate strategies like these. If you are curious about your students' thinking, you might pose one of these problems to your class. Encourage students to imagine the situation and to visualize the quantities and their relationships: Have they ever shared sandwiches or made a batch of cookies? Can they picture a pint of paint and painting a dog house? Allow students to follow their own line of reasoning, and then listen for what the strategies reveal about their understanding of fractions as quantities. The problem that Gardner selected could be solved in several different ways, and each student had a way to get started, from direct modeling of fractional quantities to combining fractions and reasoning about the inverse relationship between multiplication and division. Maggi, Gabriela, Jake, and Olivia all had something to contribute.

Common Core Connection
SMP 1

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The Richness of Children's Fraction Strategies

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms, and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs.

The following questions related to "The Richness of Children's Fraction Strategies," by Laura B. Kent, Susan B. Empson, and Lynne Nielsen, are suggested prompts to aid you in reflecting on the article and on how the authors' idea might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

Posing fraction problems using contextualized situations provides many opportunities for teachers to assess and extend students' understanding of fractional quantities and their relationships and engages them in the Common Core's eight Standards for Mathematical Practice (SMP) (CCSSI 2010). Teachers who are successful at building on children's mathematical thinking have spent time analyzing students' strategies and interacting with students one-on-one to understand their thinking. This work helps teachers learn to interpret children's strategies and the mathematical knowledge these strategies reflect.

We encourage teachers to probe students' thinking and analyze their written work to identify how strategies reflect students' knowledge of fractions. We also encourage teachers to plan problems together in teams when possible and work together to make sense of students' thinking.

Ideas to consider as you read and reflect on this article

Teachers who engage in professional development that focuses on using problem posing to elicit their students' thinking about content will typically pose one or two problems in a given class period and allot time for students to solve the problem and share their strategies with the rest of the class. How is it possible with this approach to address a variety of content and practice standards in one lesson?

How would your own students approach solving the Hot Chocolate problem if they were not told it is a division problem or given a strategy to use to get the answer? Would there be a range of approaches, or would students solve the problem using a similar method? How might you help them connect their different strategies?

The strategies reported in this article reflect children's foundational understanding of fractions as quantities and also help advance this understanding in ways that apply to other fraction concepts. The following questions could be considered as the strategies presented in this article are analyzed for mathematical connections.

- What does it mean to understand a fraction as a quantity?
- What are some examples of students working with fractions in ways that reflect a lack of understanding of a fraction as a quantity?
- How would students' understanding of fractions as quantities help them solve the Hot Chocolate problem?

We invite you to tell us how you used Reflect and Discuss as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to Teaching Children Mathematics *at* ***tcm@nctm.org***. *Please include Readers Exchange in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.*

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Laura B. Kent is an associate professor of mathematics education at the University of Arkansas. She is interested in student's mathematical strategies and classroom-based professional development. **Susan B. Empson** is a professor of STEM Education at the University of Texas–Austin. She studies teaching and learning in the elementary classroom with a focus on fractions. **Lynne Nielsen** is an assistant professor of mathematics education at Louisiana Tech University. She studies preservice and teacher PD, teacher questioning strategies, student engagement, and student thinking.