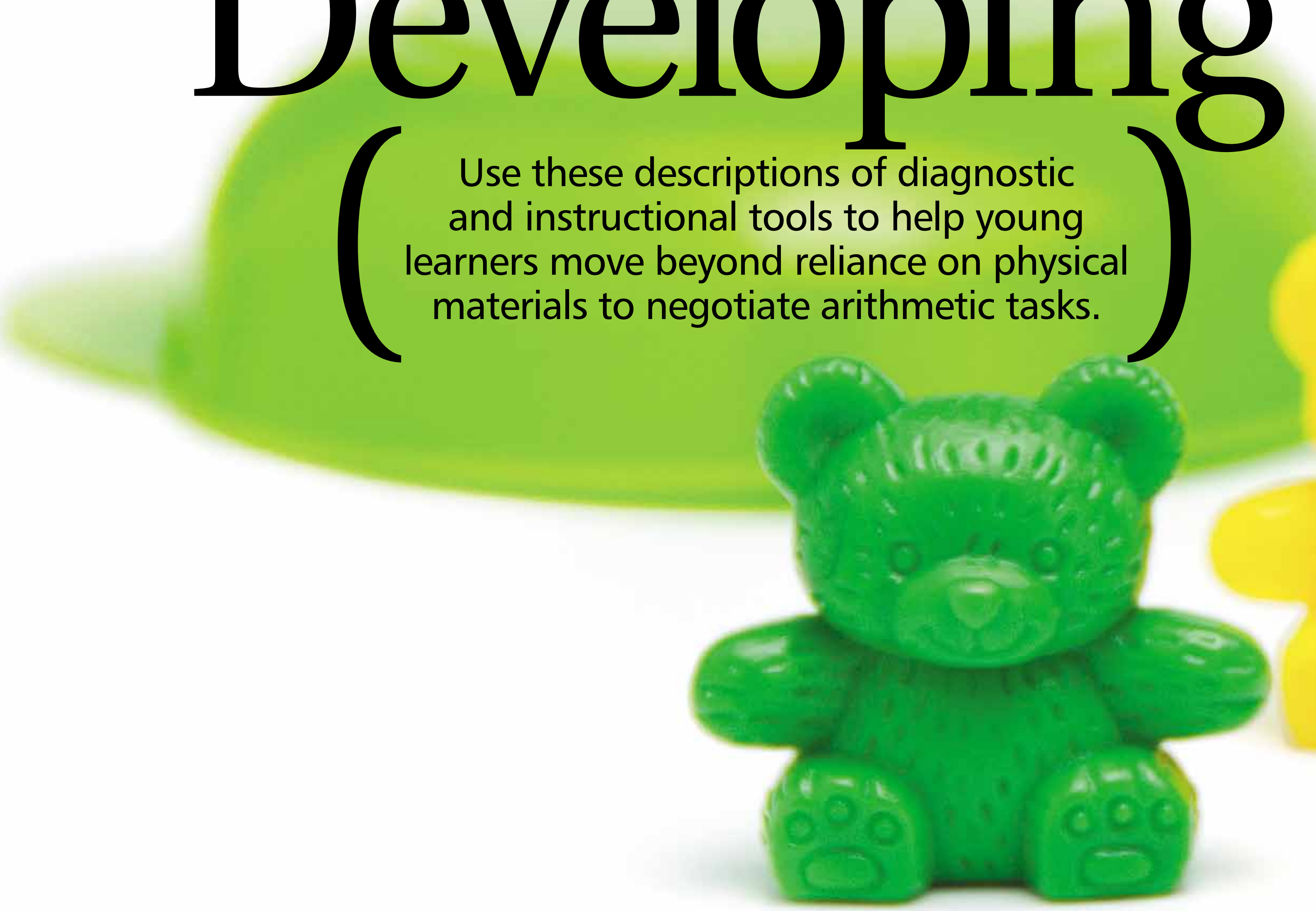
2012 FOCUS ISSUE:
Differentiation

Developing

(Use these descriptions of diagnostic and instructional tools to help young learners move beyond reliance on physical materials to negotiate arithmetic tasks.)



Joseph is a first-grade student whose teacher is working with him to better understand the manner in which he adds. Consider the following exchange around the task 11 + 4.

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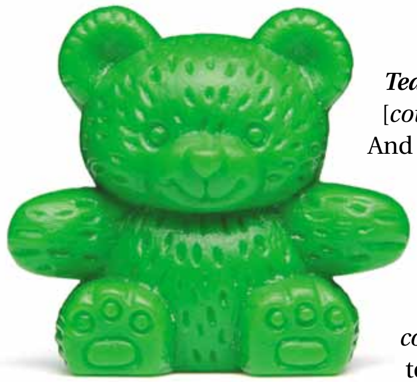
Mental

Quantitative

Imagery



> ***Teacher:*** So, there are eleven under here [*covering a collection of blue counters*]. And then we put four more with it [*covering a set of red counters*].
>
> ***Joseph:*** Six … eleven [*touching the cover on top of the red counters four times in an approximation of the same spatial pattern as the concealed counters beneath*], twelve, thirteen, fourteen, [*whispering*] fifteen. Fifteen!

Moving beyond physical interactions with materials is a significant mathematical step for students that is often difficult to take. Persistent tally-mark use, for example, among older children is a testament to this challenge. For many students, shifting away from tangible tools begins a precarious journey; teachers should support it with thoughtfully tailored instruction. Indeed, this journey begins with helping students "see" math with their "mind's eye" via the construction of quantitative mental imagery.

The vignette above shows us that Joseph is working in a somewhat novel setting. The counters are still within his physical space;

FIGURE 1

Each new phase in the Stages of Early Arithmetic Learning (SEAL) model incorporates, rather than displaces, the knowledge and understanding of prior stages.

Stages of Early Arithmetic Learning (SEAL)
(Steffe et al. 1988; Steffe 1992; Wright et al. 2006)

Emergent
Child approximates counting activity (e.g., saying number words when asked, "How many?") but is typically unable to determine the numerosity of a collection.
What it might look like: Child is presented with a collection of twelve counters and asked how many are there. Child touches some, but not all, the counters, saying, "One, two, three, five, seven, eight, nine—nine!"
Hallmark strategy: *Attempting to count a collection*

Perceptual
Child can determine the numerosity of collections when physical materials are available for counting but is unable to negotiate arithmetic tasks in the absence of physical materials.
What it might look like: Child is presented with collections of nine counters and five counters and asked, "How many altogether?" The child touches each of the counters while saying, "One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen!"
Hallmark strategy: *Physically interacting with materials to count collections*

Figurative
Child can negotiate arithmetic tasks in the absence of physical materials by generating mental imagery of past sensory experiences referred to as *re-presentations*.
What it might look like: Child is presented with a collection of nine counters and five counters; the counters are then concealed. Child is asked how many altogether. The child looks away, begins counting at one (may or may not sequentially raise fingers), and says, "One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen—fourteen!"
Hallmark strategy: *Continuing the count from one when materials are physically unavailable*

Initial number sequence
Child can negotiate arithmetic tasks in the absence of physical materials by constructing a single chunk of a number, referred to as *a numerical composite*, and then counting on from this chunk.
What it might look like: Child is presented with collections of nine and five counters, and the counters are then concealed. Child is asked how many altogether. Child counts on from nine (may or may not sequentially raise fingers) and says, "Ten, eleven, twelve, thirteen, fourteen—fourteen!"
Hallmark strategy: *Counting on when materials are physically unavailable*

Facile number sequence
Child can negotiate arithmetic tasks in the absence of physical materials by constructing multiple chunks of numbers, referred to as *abstract numerical composites*, and decomposing/recomposing these chunks.
What it might look like: Child is presented with collections of nine and five counters, and the counters are then concealed and asked how many altogether. The child responds immediately, "Fourteen—I borrowed one from the five to make ten, and then just added ten and four in my head."
Hallmark strategy: *Multiple non-count-by-ones strategies when materials are physically unavailable*

Shifting from tangible tools begins a precarious journey.

however, a thick, opaque screen that prevents sight or touch intentionally conceals them. Joseph appears quite able to negotiate this task, though, and uses a counting strategy to determine the sum of 11 + 4. The manner in which he appears to reconstruct the spatial pattern of the red counters suggests some connection to the hidden materials; but how, exactly, did he perform this strategy without actually touching the materials that were beneath the cover? The key to moving away from dependence on materials lies in the construction of quantitative mental imagery.

Quantitative aspect of number

Before discussing mental imagery and arithmetic strategies, a brief examination of number is necessary. Numbers as conceptual and cultural entities possess three distinct components: verbal (i.e., *four*), numeral (i.e., 4), and quantity (i.e., □ □ □ □) (Wright 1994; Wright, Martland, and Stafford 2006). Neurological confirmation of these different aspects may be found in Dehaene's (1992) triple-code model, which describes the location of neural activity as a function of number aspect. The model explains how different parts of the brain become active depending on the aspect of number with which one is working. Dissecting the different components that comprise numbers is important here, as the remainder of this article focuses primarily on work within the quantitative aspect.

Stages of early arithmetic learning

Resulting from a series of extensive teaching experiments, the Stages of Early Arithmetic Learning (SEAL) form a model for understanding how children come to understand quantity (Steffe 1992; Olive 2001; Wright, Martland, and Stafford 2006; Wright et al. 2006). Specifically, this model leverages varying arithmetic strategies to describe how children progress through stages (see **fig. 1**) of increasingly sophisticated quantitative understanding. This progression describes mathematical activity beginning with *emergent counting*—where children approximate counting but are unable to determine the numerosity of a single collection of materials—all the way to the *facile number sequence*—where children may enact multiple, different non-count-by-ones strategies to negotiate arithmetic tasks. For example, a student might accomplish 12 + 5 by "placing the 10 aside," adding 2 + 5, and then "replacing the 10" to arrive at a sum of 17.

Of particular importance are the perceptual and figurative stages, during which children move from working with quantity as a physical entity toward more abstract, mental construction of quantity. If perceptual counting is considered a direct sensory experience, then figurative counting may be thought of as one step removed from a direct sensory experience. At the figurative stage, children are no longer tied to working with items they

can see or touch, but rather can begin to leverage mental replays of past sensory experiences to facilitate counting acts (Clements and Sarama 2007; Rittle-Johnson, Siegler, and Alibali 2001; Steffe 1992). These mental replays have been termed *re-presentations* (as distinct from representations) to describe how children mentally re-present or re-enact a prior sensory experience to themselves (Olive 2001; Steffe 1992).

Perceptual and figurative arithmetic strategies

Jennifer and William are first graders at the same school. In the exchanges that follow, a teacher attempts to determine how each child understands and works with quantity. Consider first the exchange between Jennifer and her teacher.

Teacher: Five reds [*placing a cover over a collection of five red counters*] and four blues [*placing a cover over four blue counters*]. How many altogether [*waving a hand across both covers*]?
Jennifer: Nine [*displaying five fingers on her left hand and touching each of them with the index finger of her right hand and then displaying four fingers on her left hand and touching each of them with the index finger of her right hand*].
Teacher: Let's try this one: nine red counters [*placing a cover over a collection*] and six blues [*placing a cover over them*]. How many altogether [*waving her hand across both covers*]?

FIGURE 2

To decide on suitable next steps, teachers need a practical way to diagnose students' mathematical understanding. These SEAL instructional examples offer such diagnostic measures.

Stages of Early Arithmetic Learning (SEAL) Instructional Examples
(Wright et al. 2006)

Emergent
Key next step: **Developing the capacity to accurately count a collection of materials**
Instructional examples: (1) Present the child with a collection of twelve counters and ask, "How many counters are in this pile?"
(2) Present the child with a collection of thirty seashells and ask, "Can you get me fourteen of those seashells?"

Perceptual
Key next step: **Developing the capacity to mentally re-enact (i.e., visualize) arithmetic experiences**
Instructional examples: (1) Present the child with a collection of nine red counters and seven blue counters. Screen the second collection and say, "Nine counters and seven more under the cover. How many altogether?"
(2) Present the child with the same task, but in this instance, screen both collections.

Figurative
Key next step: **Developing strategies to count on and count back during arithmetic tasks.**
Instructional examples: (1) Present the child with a collection of twenty-seven red counters and two blue counters. Screen both collections, and say, "Twenty-seven counters and two more under the covers. How many altogether?"
(2) Present the child with twelve counters, and then screen the collection. Remove four counters from under the screen, and place them under a second screen. Say, "We started with twelve counters, but then I took four out. How many counters are left?"

Initial number sequence
Key next step: **Developing non-count-by-ones (composite) arithmetic strategies**
Instructional examples: (1) Present the child with two 10-bundles of Popsicle sticks and four loose sticks and then screen them. Ask, "How many sticks are under the cover?" Add additional 10-bundles and single sticks, and ask, "How many now?" after each addition.
(2) Present the child with five 10-bundles of Popsicle sticks and three loose sticks, and then screen them. Ask, "How many more sticks do I need to have sixty sticks? How about to have seventy sticks?"

Facile number sequence
Key next step: **Extending composite addition and subtraction strategies**
Instructional example: Task the child with mentally solving two-digit and three-digit addition and subtraction tasks. Ask how he or she worked out the problem. Model the child's strategy with an empty number line (if the child uses a jump strategy, e.g., 36–24: "I went backwards twenty and landed at sixteen, and then four more to twelve") or a tree diagram (if the child uses a split strategy, e.g., "I took twenty from thirty and got ten. Then I took four from six and got two, so it's twelve").

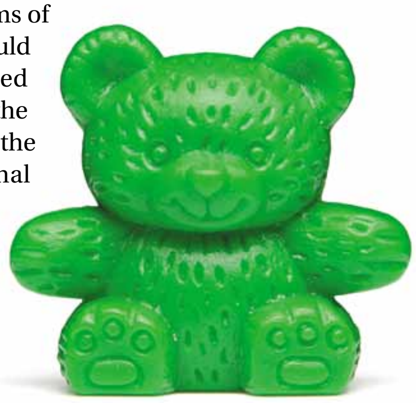
Jennifer: Six? [*displaying five fingers on her left hand and touching each of them with the index finger of her right hand and then displaying four fingers on her left hand and touching each of them with the index finger of her right hand*]
Teacher: OK, how did you know?
Jennifer: I counted my fingers.

Jennifer is able to negotiate arithmetic tasks where the materials have been concealed; however, her strategy involves substituting the concealed materials with her own fingers. As one might suspect, when a task's sum is greater than ten, Jennifer is unable to determine the numerosity of the two collections—quite simply, she runs out of fingers. Interestingly, in similar cases, children will sometimes attempt to count other accessible, physical items (e.g., toes, bricks of a classroom wall, ceiling tiles, etc.). The key point here is that Jennifer's strategy appears *perceptual*; she relies on physical interaction with materials (her fingers) to operate arithmetically.

In another classroom, William is working on the same quantitative tasks. Consider the following exchange between William and his teacher.

Teacher: OK, nine chips right there [*placing a cover over a collection of counters*] and six chips [*placing a cover over blue counters*] How many altogether [*waving her hand across both covers*]?
William: [*raising nine fingers sequentially and whispering*] One, two, three, four, five, six, seven, eight, nine, ten. [*Looking away from the covers and shutting his eyes, he lowers his fingers on one hand, raises them again sequentially, and whispers.*] Eleven, twelve, thirteen, fourteen, fifteen. Fifteen.

William has a strategy for thinking arithmetically in the absence of physical materials. Specifically, he does not need fifteen countable objects to negotiate the task (9 + 6) but can re-present involved quantities to arrive at a solution. Both Jennifer and William used their fingers during their mathematical activity; however, Jennifer appeared to consider her fingers as physical objects to be counted, whereas William raised his fingers sequentially, ostensibly to help him keep track of his *figural re-presentation*. Indeed, key to this exchange is William's capacity, via mental imagery, to move beyond physical interactions with materials in the context of arithmetic tasks. In terms of instructional next steps, Jennifer could likely capitalize on partially screened arithmetic tasks (e.g., 9 + 6, but only the second addend [6] is screened) to aid the development of her re-presentational capacity. William, already apparently re-presenting with facility, would likely benefit from tasks aimed at furthering this capacity into counting-on strategies. For example, with William, we might pose the task 32 + 3 with fully screened counters. The disparity in the two addends is designed to help William curtail his count-from-one approach. The first addend is inconveniently large, and the second addend is tantalizingly small (see **fig. 2**).



Making practical diagnoses

Mathematical teaching and learning is most effective when the teacher is able to enact the right task for the right child at the right time. This necessarily means that some manner of diagnosis is necessary to determine appropriate instruction. Although mathematics intervention specialists may wish to use robust, diagnostic interviews (Wright et al. 2006) to precisely ascertain students' mathematical understanding, classroom teachers often need measures that are more expedient. Ideally, cognitive determinations of a child are based on observations across a range of activities, but some relatively brief task progressions can help teachers distinguish between children's perceptual and figurative arithmetic strategies. Jennifer's and William's earlier tasks (see **fig. 3**) are a good way to differentiate between the strategies. As the reader observed with Jennifer, children who have yet to develop figurative arithmetic strategies will typically be unable to negotiate the third and final task in the diagnostic progression above. Thus, teachers may then tailor instruction to help these students begin to move beyond physical interactions with materials.

Specific tasks to support the development of quantitative mental imagery

Returning to the notion that effective mathematical tasks are those tailored to the individual child, helping students advance their mathematical thinking and strategies beyond physical interactions requires specific instructional tactics aimed at fostering the development of mental imagery and representational capacity. Typically, this instruction features physical materials of some kind; however, to support imagery construction, the materials are often presented and then concealed. Consider this exchange with Jennifer involving random objects concealed by an opaque red screen—referred to as a linear imaging task (see **fig. 4**).

Teacher: Ready? [*She places the screen over a linear arrangement of four objects from left to right: a car, a bear, a rooster, and a frog.*] What's on this end [*touching the far right-hand side of the screen*]?
Jennifer: [*pausing for eight seconds*] Frog. How many things came before that frog [*dragging her finger across the screen from right to left*]? I think I put five things [*touching the screen three times in a linear pattern from right to left*].
Teacher: You think so? You want to have another look [*raising the screen*]?
Jennifer: One, two, three, four, oh.
Teacher: So [*lowering the screen*], there is a frog [*touching the far right-hand side of the screen*]. How many things came before the frog [*dragging her finger across the screen from right to left*]?
Jennifer: [*looking across the table toward a shell and a block placed in front of the teacher and touching the screen in a linear pattern from right to left*] Rooster, shell … Oh! I didn't get the shell [*motioning toward the shell in front of the teacher*]. Rooster, bear [*touching the screen two times in a linear pattern, from right to left*], car, three [*rapidly raising three fingers sequentially*].
Teacher: Three things came before the frog.

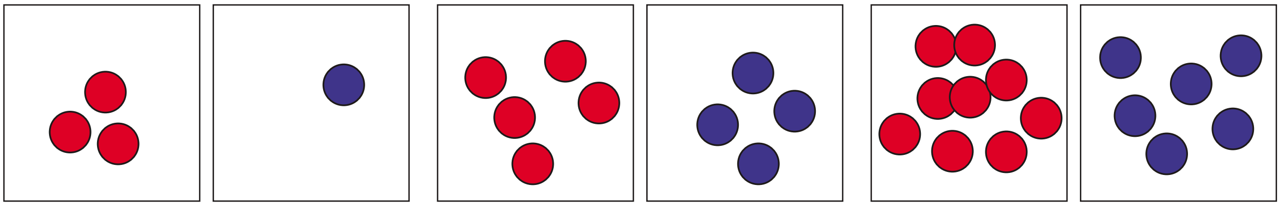
First, notice the manner in which the teacher allows Jennifer the opportunity to view the materials again during the task; however, the teacher replaces the screen during the questioning portions of the task. Note also the manner in which Jennifer touches the top of the screen (in a linear pattern), suggesting a productive connection between her thinking and the tool. Again, the intent with this presentation is to develop Jennifer's incipient capacity to re-present. Prior to this work, Jennifer appeared unable to move beyond perceptual arithmetic strategies—specifically those involving finger patterns.

FIGURE 3

Students who have not yet developed figurative arithmetic tactics will typically be unable to negotiate the last step of task 2 in the diagnostic progression.

Perceptual vs. Figurative Diagnostic Task Progression

Task 1	Task 2	Task 3
3+1	5+4	9+6
Presentation:	Presentation:	Presentation:
Present three red counters, and then conceal them; present one blue counter, and then conceal it.	Present five red counters, and then conceal them; present four blue counters, and then conceal them.	Present nine red counters, and then conceal them; present six blue counters, and then conceal them.



Note: Children should not be able to see or touch counters when they are concealed.

FIGURE 4

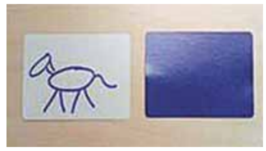
To develop Jennifer's capacity to re-present, the teacher gave her a chance to view the materials again during the linear imaging task but replaced the screen when questioning the child.

FIGURE 5

After Jennifer commented on the difficulty of this row task, the teacher micro-adjusted it so the child could physically interact with the cards, giving her confidence in her solution.

Animal Card Task

Teacher: What if I had three animal cards in a stack [*fanning a stack of three animal cards facedown and then restacking*]; how many animal legs do you think I have?



Jennifer: That's a hard one!
Teacher: Is there any way you can figure it out?
Jennifer: No, I can't do it [*shaking her head*].
Teacher: If I put them out like this [*placing the three cards in a row,* ***facedown*** *in front of Jennifer*] does that help?
Jennifer: [*nods*]
Teacher: How does that help you?
Jennifer: One, two, three, four [*touching the back of each card four times in the approximate location of the legs*], five, six, seven, eight, nine, ten, eleven, **twelve**.
Teacher: OK, twelve. Do you need to check it?
Jennifer: No [*shaking her head*].

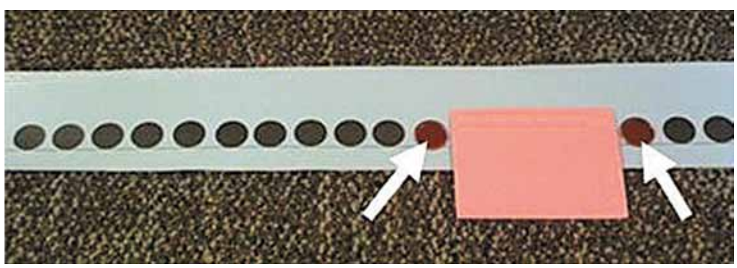
Teachers might use many other tools to help students develop quantitative mental imagery. Consider Jennifer's work with a linear arrangement of dots—referred to as a *row task*—and an arrangement of animal cards (see **fig. 5** and **fig. 6**). Similar to the linear imaging task, negotiating the row task involves re-presenting concealed objects; however, in this instance, the screened quantities are uniform in appearance and do not have any unique or distinguishing characteristics (e.g., a yellow car, a green frog, etc.). Turning to Jennifer's work with the animal cards, the use of "four-legged animals" is a fairly effective support in that the screened quantities are now grouped into more manageable and natural chunks. Also, note the manner in which this task was initially presented (as cards in a stack). After Jennifer remarked that this task is "a hard one" and that she could not work it out, the teacher adjusted the task so that Jennifer was able to physically interact with the cards, although they were facedown. Interestingly, Jennifer did not appear to need to see the images of the animals to count the legs, but she did seem to need markers (i.e., the facedown cards) for each animal to help her keep track of her re-presentation. This

FIGURE 6

This row task involves re-presenting concealed objects that are uniform in appearance and have no unique or distinguishing characteristics.

Row Task

Teacher: So let's see here; there is eleven [*placing a translucent red counter on top of the eleventh dot on the row tool*]. And I am going to cover up four [*placing a screen over the twelfth, thirteenth, fourteenth, and fifteenth dots on the row tool*]. What is this guy [*placing a translucent red counter on top of the sixteenth dot on the row tool*]?
Jennifer: Twelve, thirteen, fourteen, fifteen [*touching the screen four times in a linear pattern and audibly whispering the numbers*]. You covered up fifteen, and then that's sixteen [*touching the counter on top of the sixteenth dot on the row tool*].



modification certainly reduces the level of demand for this particular task (i.e., the introduction of perceptual markers for each animal card). However, this new task arguably presents a more meaningful and connected transition point toward work with increasingly sophisticated figurative experiences: Note the child's apparent confidence in her solution. Indeed, after several tasks where the cards are presented individually (facedown), the teacher might return to pose variations of the original task, where the cards are presented in a stack. The point is that working to help children advance their thinking and strategies in this area will often require teachers to make micro-adjustments to tools and task presentations as they are teaching, to increase task accessibility. Some potential modifications include the following:

- Increasing or decreasing the concealed quantities
- When working with multiple quantities, screening only some of them (e.g., presenting animal cards both faceup and facedown)
- Incorporating color into concealed quantities (e.g., using different color counters to denote the two addends in an addition task)
- Adding structure or patterns to concealed quantities (e.g., arranging collections of counters in domino patterns before concealing them)

Supporting imagery development

Although individual interactions may be productive means to help children develop quantitative mental imagery, such interactions are often impractical for classroom teachers; thus, teachers may elect to work with small groups of children who have all demonstrated strong perceptual counting strategies but seem unable to move beyond physical interactions with materials. Here, game contexts frequently prove useful. For example, the teacher might distribute a single six-sided dot die to each of the children and might keep two dice for herself. She rolls the two dice, announces the number, and then conceals the dice with a cover. Going around the table, each child rolls his or her die and announces the sum of the child's die plus the teacher's dice. The child with the largest sum wins the round. This game approximates partially screened arithmetic tasks that are useful in helping children transition from perceptual to figurative counting. Additionally, the use of three dice increases the probability that sums will be beyond finger-counting range.

Supporting the development of quantitative mental imagery

Adapting curriculum materials to support the development of quantitative mental imagery can be as simple as adding a cover to an existing task. For example, if the task involves children adding sections of train cars to determine the entire length of a train, one section might be inside a small tunnel obscuring two or three of the cars. With tasks involving pictorial representations of addition tasks, the picture of the smaller addend could be covered with a sticky note, preventing students from readily counting each perceptual item in the task. However, if needed, the covers can easily be removed to support the development of the quantitative mental image.

From materials to mind's eye

Given the aim for children to construct an increasingly abstract understanding of mathematics, robust support for mental imagery and re-presentation is of considerable importance. Even a brief series of tasks can afford teachers diagnostic power to design tailored and effective instruction that helps a child transition from materials to mind's eye.

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