Using Covariation Reasoning to Support Mathematical Modeling

Table representations of functions allow students to compare rows as well as values in the same row.

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For many students, making connections between mathematical ideas and the real world is one of the most intriguing and rewarding aspects of the study of mathematics. In the Common Core State Standards for Mathematics (CCSSI 2010), mathematical modeling is highlighted as a mathematical practice standard for all grades. To engage in mathematical modeling, beginning algebra students must learn to use their understanding of arithmetic operations to make mathematical sense of problem situations and to relate this sense making to functions represented by equations, tables, and graphs. The word problems commonly used in beginning algebra courses give opportunities to practice mathematical modeling. Further, the ability to reason with quantities as well as numbers is an important capacity for students to develop.

Quantities are conceptions of things that can be measured, such as distance or time. The measure of a quantity has a defined unit and a process for assigning a number that represents the proportional relationship between a particular value of the quantity and the unit (Thompson 2011). Quantitative reasoning is different from numerical reasoning because quantitative reasoning involves a clear mental image of how quantities are related (Thompson 2011). Someone who quantitatively understands the average speed of a sprinter who runs 100 meters in 4 seconds might imagine the 100-meter track divided into 4 sections, each 25 meters long, and then imagine the runner traversing one section during each second. By contrast, if students compute 100 m/4 s = 25 m/s, they may understand only the arithmetic relationship between the numbers 100, 4, and 25 without understanding why dividing 100 by 4 makes sense in this situation. Quantitative reasoning is a key resource for students who are learning to use algebra to model relationships between quantities that vary.

Two kinds of quantitative reasoning have a special relevance for beginning algebra students. The *correspondence* perspective deals with the question, How is one quantity related to another? A correspondence understanding of speed might be expressed as the rule that relates each value for time with a unique value for distance, such as the equation $y = 25x$, where x represents time and y represents distance. By contrast, the key question for *covariation* reasoning is, How does one quantity change as another quantity changes? A covariation understanding of speed would focus on how distance and time change together—that is, the distance covered increases by 25 meters as the elapsed time increases by 1 second.

Both kinds of reasoning are important goals for algebra students. Correspondence is a fundamental piece of mature reasoning about functions, and covariation is critical for developing the rate-of-change concept. Research shows that covariation is a common entry point into algebra for students (e.g., Confrey and Smith 1995); but traditional approaches to teaching algebra emphasize correspondence and often have little or no treatment of covariation (Smith 2003). Thus, this article focuses on students' use of covariation and how to support it the classroom.

Presented here are two sessions from Ms. Holmes's classroom (the teacher's name is a pseudonym) in which seventh graders intuitively used covariation to begin to make sense of word problems. These passages show how students' covariation reasoning might surface in the classroom and illustrate some of the teaching strategies that Ms. Holmes used to support her students' reasoning. The sessions also provide a foundation for the discussion of classroom strategies, which summarizes research-based strategies for supporting students' use of covariation reasoning to build robust mathematical models.

The teacher's sequence of questions implied a comparison between the paired quantities. The student's responses indicate a comparison between rows in the hypothetical table of values.

CLASSROOM SESSION 1

In session 1, a small group of students was working on problem 1 during a lesson spent reviewing for a state achievement test.

Problem 1: Sally needs 3 2/7 yards of fringe to trim each drape. If she has 8 drapes, how much fringe does she need? What operation is used to solve this problem?

(a) addition
(b) subtraction
(c) multiplication
(d) division

The first student started by guessing that the operation was subtraction but after some time changed his mind.

***Student 1*:** I don't think you subtract now.
***Ms. Holmes*:** What do you [*other*] guys think?
***Student 2*:** Um . . . divide?
***Student 3*:** Yeah.
***Ms. Holmes*:** Divide? Why?
***Student 2*:** Because . . . [*thirty-five seconds elapse*] . . . I don't know.

Rather than challenging student 1, Ms. Holmes asked for other students' ideas. She also pressed students to justify their answer of division.

To help the students make progress, Ms. Holmes next read the problem out loud, asked the students to explain the problem in their own words, and had them draw a picture of the situation. After a few minutes, the students were still stuck, so she asked a sequence of questions:

***Ms. Holmes*:** So, if Sally had 1 drape, how many yards would she need?
***Student 1*:** Three and two-sevenths [3 2/7] . . .
***Ms. Holmes*:** All right. What if she had 2?
***Student 1*:** She would need . . . twice that 3 2/7.
***Ms. Holmes*:** [*nods*] What if she had 3?
***Student 1*:** Twice . . . I mean, 3 times that.
***Ms. Holmes*:** So, what are you doing each time?
***Student 1*:** Multiplying . . . Oh! . . . So you multiply. It's multiplication.

Ms. Holmes then asked the two other students to explain why multiplication was the appropriate operation.

Discussion of Session 1

Each of Ms. Holmes's questions asked student 1 to compare a quantity of drapes with the corresponding quantity of fringe, so each of these questions is about correspondence. Neither Ms. Holmes nor the students used a table of values for this problem, but it is natural to imagine a hypothetical table (see **fig. 1**). In such a table, each of Ms. Holmes's questions asks about values within a single row.

What is interesting about this episode is that the sequence of questions implied a comparison between the paired quantities. The implicit reasoning supporting student 1's statement that 2 drapes would need "twice that 3 2/7s" is that 2 drapes are twice as many as 1 drape. The student's responses indicate that he recognized that as the quantity of drapes doubles and triples, the quantity of fringe is multiplied by 2 and then by 3. This reasoning can be understood as a comparison between rows in the hypothetical table of values (see **fig. 1**).

The final part of the episode involved looking back over the sequence of examples. Ms. Holmes's final question was probably intended as a general correspondence question, with "each time" referring to each correspondence between the number of drapes and the amount of fringe. The student likely interpreted "each time" to refer to each new pair of drapes and fringe and compared the new pair with the initial one. In any event, by asking a sequence of specific questions and then asking the student to reflect across these examples, Ms. Holmes was able to build this student's covariation reasoning and help him establish a meaningful mathematical model of the quantities in problem 1.

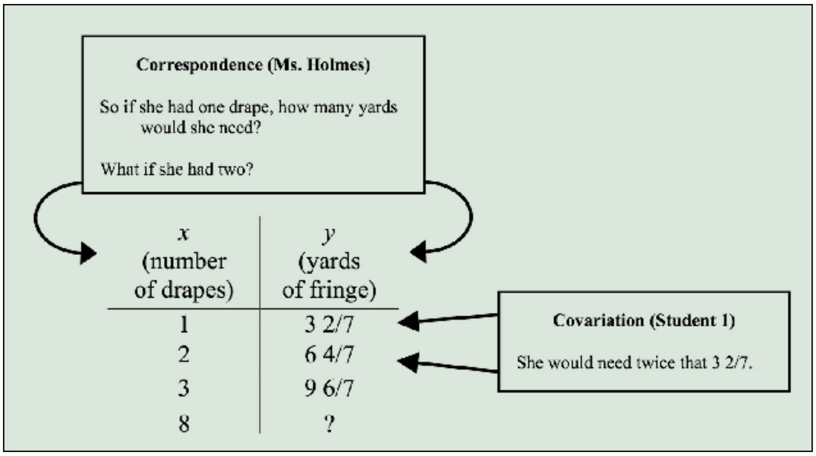

Fig. 1 The teacher intends to build a correspondence between quantities in the same row. The student sees a covariation between rows.

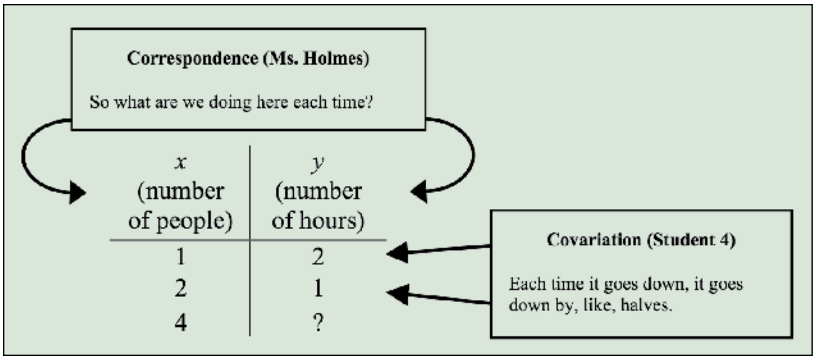

Fig. 2 The student generalizes (incorrectly) using covariation.

CLASSROOM SESSION 2

The class discussion of problem 2 occurred on the same day but during a different period and with different students. Earlier in the year, these students had worked on distinguishing directly and inversely proportional relationships and on writing linear equations using tables of corresponding x- and y-values.

Problem 2: Juan can clean up after the party in 2 hours if he works alone, but he hopes his friends will help. Write an equation relating the number of people (x) and the amount of time (y) it would take to clean up if everyone works at the same rate.

One student initially guessed that the equation was $y = 2x$. However, Ms. Holmes pointed out that this equation would not work because as the number of people increased, the amount of time should decrease. The students were still stuck, so to direct the students' reasoning about the problem, Ms. Holmes constructed a table on the whiteboard that included three values for x (the number of people): 1, 2, and 4 (see **fig. 2**). As she questioned the students, they readily agreed that one person would take 2 hours and that 2 people would take just 1 hour, but they disagreed about the time it would take 4 people.

***Ms. Holmes*:** What if you had 4 people?
***Student 1*:** Thirty [30] minutes.
Ms. Holmes: How are you getting that?
***Student 1*:** It's . . . um . . .
***Student 2*:** It'd be 15 minutes . . . because it's half of . . . because if you have 3 people, it would be 30, and 4 it would be 15.
***Ms. Holmes*:** Explain to me what you're thinking.
***Student 3*:** x divided by two equals y.
Ms. Holmes*:** x divided by 2? [*points to the first row of the table in* ***fig. 2] Well, 1 divided by 2 is 1/2 . . .
***Student 3*:** Oh, it's y divided by 2 equals x!
***Ms. Holmes*:** Well, 2 divided by 1 is 2, but 2 divided by 2 . . .
***Student 4*:** [*interrupting*] Each time it goes down, it goes down by, like, halves.
***Ms. Holmes*:** So, what are we doing here each time [*pointing to x and then y in the table*]?
***Student 5*:** Going down by a half each time.

Only a few minutes were left in the period, so Ms. Holmes went on to remind the class of the equation $y = k/x$ (the correspondence rule). She showed that this rule was algebraically equivalent to $yx = k$. Finally, she demonstrated that the product of x and y was always 2 in this problem.

Discussion of Session 2

In this episode, Ms. Holmes pressed students to explain their thinking and offered counterexamples when students guessed incorrectly. In this way, she enabled students to describe their thinking and also promoted mathematical reasoning.

Just as in classroom session 1, the key was Ms. Holmes's sequence of questions about correspondence. As before, several students were reasoning about covariation in the context of the problem. Student 2 hypothesized that the value for 3 people was 1/2 hour, or 30 minutes, and his hypothesized time for 4 people was half the previous value—1/4 hour, or 15 minutes—perhaps because he believed that each new person would cut the time in half. Student 4 apparently picked up on this idea and made a more general covariance statement.

The final part of the episode involved looking back over the sequence of examples. Ms. Holmes's final question was intended as a general correspondence question, just like her final question of the first episode. Her gestures indicated that "each time" referred to each row relating a specific number of people and the corresponding amount of time. However, student 5 reasoned about covariance instead of correspondence and interpreted "each time" to mean each new row. He compared each new row with the previous one, claiming that time was "going down by a half each time." Unfortunately, the period ended before these students had an opportunity to differentiate correspondence and covariation reasoning about the problem situation or find the correct mathematical model.

Developing the ability to clearly and explicitly reason about quantities that are changing together will support students in beginning algebra and lay the foundation for later success in mathematics.

STRATEGIES FOR THE CLASSROOM

As we review the classroom episodes and the research literature (e.g., Carlson and Oehrtman 2005; Ellis 2011), several strategies emerge that teachers can use to support students' covariation reasoning:

1. Use a sequence of specific pairs of values to support students' reasoning about the problem.
2. Ask students to describe and explain their thinking about a single pair of values and to compare different pairs of values. Listen carefully for covariation or correspondence reasoning.
3. If students make incorrect claims, ask for other students' ideas. Model quantitative reasoning by providing a mathematical reason or counterexample based in the problem situation that explains why the claim is incorrect.
4. Support covariation reasoning by asking students the following kinds of questions about problem situations (adapted from Carlson and Oehrtman 2005):

- What quantities are changing together, and how are they changing?
- As one quantity increases, does the other quantity increase or decrease?
- As one quantity increases in constant increments, by what amount does the other quantity change?
- As one quantity increases in constant increments, what is the rate of change in the other quantity?

Selecting specific examples (strategy 1) so that one quantity is changing in constant increments might emphasize covariation relationships for students, such as Ms. Holmes's choice of 1, 2, and 3 for problem 1. On the other hand, the choice of a doubling sequence 1, 2, and 4 for problem 2 may have contributed to the students' misunderstanding of the relationship between people and time.

As teachers listen carefully and help students communicate clearly about covariation, there will be opportunities for mathematical exploration that can enrich students' conceptions of functions. For example, students 2, 4, and 5 in episode 2 were actually describing a different function that could be represented with the correspondence rule, $y = 2(1/2)^{x-1}$, or in terms of covariation (y decreases by a factor of 1/2 as x increases by 1). A teacher might ask students in the class to compare this function with the correct function for problem 2: $y = 2/x$.

REASONING ABOUT COVARIATION IS CRUCIAL

Reasoning about covariation is not a crutch; it is a crucial skill. Developing the ability to clearly and explicitly reason about quantities that are changing together will support students in beginning algebra and lay the foundation for later success in mathematics. Students who intuitively use covariation to think about how quantities change may not have access to simple, unambiguous language to describe their thinking. When using tables to develop students' understanding of functions, teachers can help students describe, explain, compare, and relate covariation reasoning between rows and correspondence reasoning within rows.

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