Reconcil

One student's unexpected solution led us to make sense of the problem about random triangles and persevere to a solution.

ing REPRESENTATIONS

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Sometimes a student's unexpected solution turns a routine classroom task into a real problem, one that the teacher cannot resolve right away. Although not knowing the answer can be uncomfortable for a teacher, these moments of uncertainty are also an opportunity to model authentic problem solving. This article describes such a moment in my class Problem Solving for Teachers. It started when I (coauthor Zahner) introduced this calendar problem from *Mathematics Teacher*:

> Triangles with sides (a, b, c) are randomly generated in the following manner: $c = 1$, $0 < a \leq 1$, and $0 < b \leq 1$. Any value of (a, b, c) that does not satisfy the triangle inequality theorem, $a + b > c$, is discarded. What is the probability (to the nearest hundredth) that a random triangle is obtuse? (McLoughlin 2002 p. 30)

I found this problem fun, and constructing the solution led me to make surprising connections among algebra, geometry, and probability. Through class problem solving and discussion of solutions (following principles from Smith and Stein 2011), I planned to connect the geometry and probability standards in the Common Core State Standards for Mathematics (CCSSI 2010) with the NCTM Standards and the Standards for Mathematical Practice (SMPs).

My plans changed when two students created different representations that appeared to result in contradictory solutions. Reconciling these solutions led my students and me on a multiweek quest to understand what was going on. Ultimately, this process of inquiry was a powerful lesson about the problem-solving practices as envisioned in the NCTM's Standards (2000) and in the Common Core State Standards for Mathematics (CCSSI 2010).

Coauthor Dent, then a mathematics teacher at Boston University Academy and a student in Calculus for Teachers, reconciled the solutions using multivariable calculus, with help from one of his high school students. Here we share the story of this problem, reflect on lessons that we learned about teaching through problem solving, and discuss pedagogical strategies that helped us navigate uncertainty in our mathematics classrooms.

AN EXPECTED SOLUTION

In planning to teach this problem, I solved it by choosing trial values for a and b and looking for patterns (indicating a nice connection with the SMP 8: "Look for and express regularity in repeated reasoning"). For example, if $a = 0.7$ and

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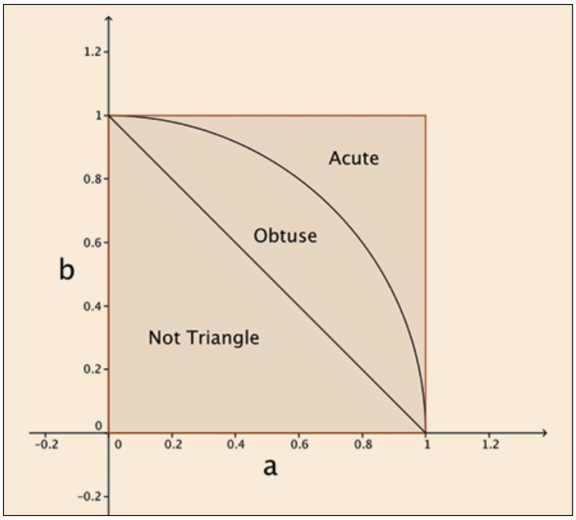

Fig. 1 A graphical representation shows possible pairings of *a* and *b* with regions indicating the type of triangle formed.

a\b	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.02	0.05	0.1	0.17	0.26	0.37	0.5	0.65	0.82	1.01
0.2	0.05	0.08	0.13	0.2	0.29	0.4	0.53	0.68	0.85	1.04
0.3	0.1	0.13	0.18	0.25	0.34	0.45	0.58	0.73	0.9	1.09
0.4	0.17	0.2	0.25	0.32	0.41	0.52	0.65	0.8	0.97	1.16
0.5	0.26	0.29	0.34	0.41	0.5	0.61	0.74	0.89	1.06	1.25
0.6	0.37	0.4	0.45	0.52	0.61	0.72	0.85	1	1.17	1.36
0.7	0.5	0.53	0.58	0.65	0.74	0.85	0.98	1.13	1.3	1.49
0.8	0.65	0.68	0.73	0.8	0.89	1	1.13	1.28	1.45	1.64
0.9	0.82	0.85	0.9	0.97	1.06	1.17	1.3	1.45	1.62	1.81
1	1.01	1.04	1.09	1.16	1.25	1.36	1.49	1.64	1.81	2

Fig. 2 Calculated values of $a^2 + b^2$ are used to determine the type of triangle formed.

$b = 0.5$, then the triangle has side lengths 0.7, 0.5, and 1. Using the converse of the Pythagorean theorem, this triangle is obtuse because $0.7^2 + 0.5^2 = 0.74 < 1^2$.

To generalize, I needed to represent all possible solutions. One possibility would be to graph points with coordinates (a, b) in the unit square. Right

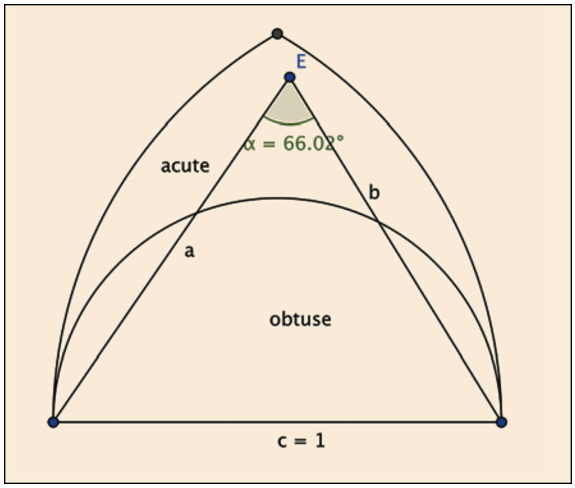

Fig. 4 Regions for the location of point *E* are associated with the type of triangle formed.

triangles correspond to points on the quarter circle $a^2 + b^2 = 1$ in the first quadrant. The restriction $a + b > 1$ excludes points in the triangle with vertices (0, 0), (1, 0), and (0, 1). Finally, the converse of the Pythagorean theorem suggests that, if $a^2 + b^2 < 1$, then the triangle is obtuse; and if $a^2 + b^2 > 1$, then the triangle is acute. Combining this information yields the graph shown in **figure 1**.

The region of nontriangles has an area of 1/2. Excluding nontriangles, the probabilities of forming obtuse and acute triangles can be found by dividing the area of each region by 1/2. Interestingly, the probability of forming a right triangle is zero, even though infinitely many right triangles are possible. One way to understand this paradoxical result is to imagine fixing *a* and asking, given *a*, what is the probability that a randomly chosen value for *b* will result in a right triangle? For any given value of *a* (e.g., 1/2), there is exactly one value of *b* that will form a right triangle (in this case, $\sqrt{3}/2$). However, the probability that *b*, a continuous random variable, takes on a particular value in its domain is zero. Since this is true for all *a* in $0 < a \leq 1$, the

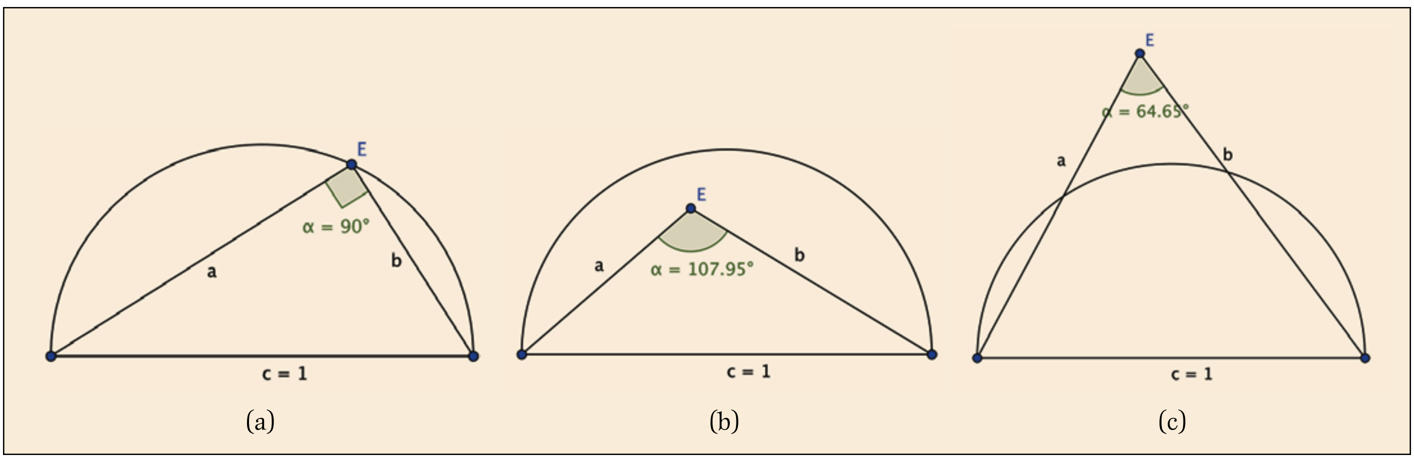

Fig. 3 Claudine's representation shows three distinct possibilities: a right triangle (**a**); an obtuse triangle (**b**); and an acute triangle (**c**).

probability that random choices of a and b will result in a right triangle is zero.

The area of each region in **figure 1** follows:

$$Area(Not\ Triangle) = 1/2 = 0.50$$
$$Area(Obtuse) = \pi/4 - 1/2 \approx 0.29$$
$$Area(Acute) = 1 - \pi/4 \approx 0.21$$
$$Area(Right) = 0$$

Therefore, to answer the problem, the probability of generating an obtuse triangle is

$$\frac{(\pi/4 - 1/2)}{1/2} \approx 0.57.$$

ALTERNATIVE SOLUTIONS

After I introduced this problem, my students solved it in groups. Most groups started by trying a few values of a and b to explore relationships, and two groups created a table showing $a^2 + b^2$ for different values of a and b. **Figure 2** shows one sample table in which the cells are color-coded by the type of triangle generated for each combination of a and b (red for nontriangles, blue for obtuse triangles, green for acute triangles, and yellow for right triangles). I encouraged these groups to consider what would happen on the borders where the color changed (e.g., when $a = 0.357$ and $b = 0.907$) to shift from the discrete case to the continuous case. The color-coded table of **figure 2** suggests the graph in **figure 1** (after a flip). Therefore, during the whole-class discussion of this problem, I planned to ask one student to share the table to discuss how organizing repeated calculations can help solve this problem.

As I anticipated, several groups made a diagram similar to that shown in **figure 1**. One student who made this solution was Katie, and I planned to ask her to share her solution next.

In addition, one group constructed a representation that I had not anticipated. Claudine recalled that any triangle inscribed in a semicircle is a right triangle. Using this fact, she drew a one-unit segment (for the side with length $c = 1$) and constructed a semicircle using c as a diameter. Finally, she constructed a triangle with side lengths a and b built on opposite ends of the segment with length c; she called the intersection of these two segments point E. Drawing a few sample triangles, Claudine reasoned that any triangle in which E landed on the semicircle would be a right triangle (see **fig. 3a**), any triangle with E inside the semicircle would be obtuse (see **fig. 3b**), and any triangle in which E was above the semicircle would be acute (see **fig. 3c**). Nontriangles did not appear in Claudine's representation.

Claudine's group mates helped find bounds for the possible location of E. The maximum lengths of a and b are 1, so the students constructed arcs of length 1 on the ends of segment c to set bounds for the vertex point E (essentially following the process for constructing an equilateral triangle). The result is an arch with an inscribed semicircle (see **fig. 4**).

Once I understood Claudine's diagram, I appreciated how it connected with the geometry of the random triangles. I asked her group to find the probabilities, and I walked away to help another group. To find the probabilities, Claudine and her group mates also calculated the area of each region, a more challenging task than finding the areas in **figure 1**. The area of the semicircle in which obtuse triangles are formed is

$$Area = \frac{\pi r^2}{2} = \frac{\pi(1/2)^2}{2} \approx 0.39.$$

The first step for finding the area of the region in which point E lands for acute triangles is to find the area of the arch. The arch is made of an equilateral triangle (region 1 in **fig. 5**) and two segments of a circle (regions 2 and 3 in **fig. 5**). The equilateral triangle's area is $\sqrt{3}/4$. The area of each segment of a circle can be found by computing 1/6 of the area of the circle and subtracting the area of the triangle. Putting these pieces together, Claudine's group found that

$$Area\ of\ arch = \left(\frac{1}{6}\pi - \frac{\sqrt{3}}{4}\right) + \left(\frac{1}{6}\pi - \frac{\sqrt{3}}{4}\right) + \frac{\sqrt{3}}{4}$$
$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} \approx 0.61.$$

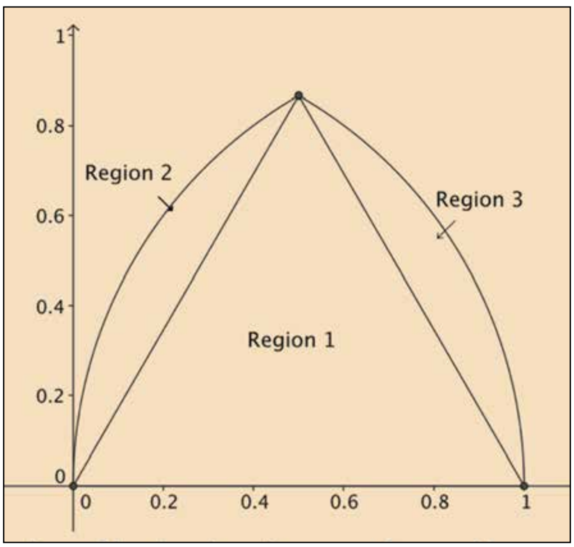

Fig. 5 The arch in Claudine's solution comprises an equilateral triangle (region 1) and two segments of a circle (regions 2 and 3).

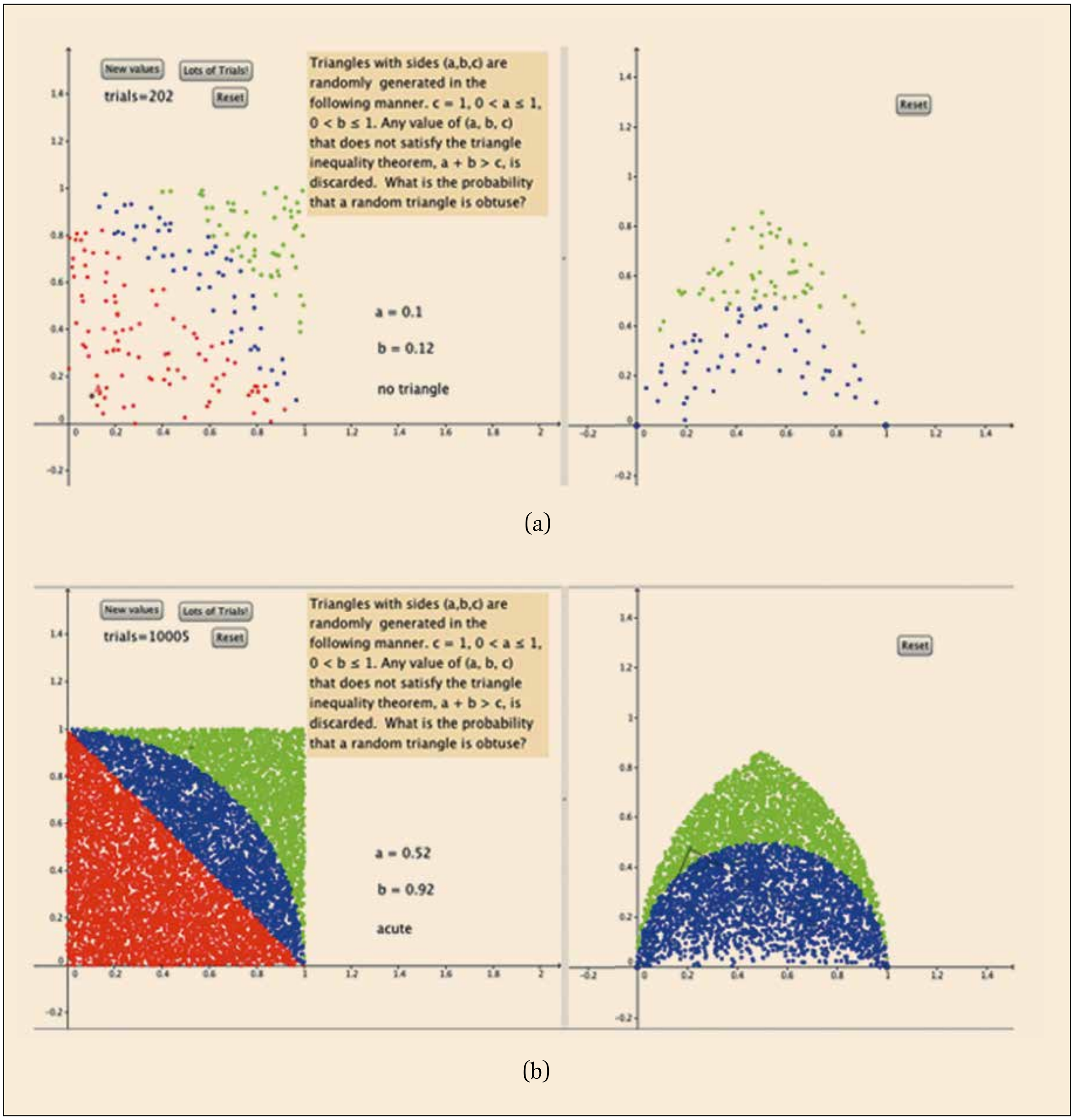

Fig. 6 The GeoGebra simulation tracks vertices *A* and *E* for 202 trials (**a**) and 10,005 trials (**b**).

Finally, they found the area of the "acute region" in **figure 4** by subtracting the area of the semicircle: 0.61 – 0.39 = 0.22.

THE UH-OH MOMENT

Gathering the class together, I asked Tyler, Katie, and Claudine to share their solutions. I was delighted that Claudine proposed to share her solution because it gave us three alternative representations to compare: a numerical table, a graph, and a geometric representation. I planned to sequence the discussion to highlight these differences.

First, Tyler shared his group's table. He discussed the fact that the table representation helped his group make sense of the problem but did not necessarily yield exact probabilities.

Next, Katie showed her solution (see **fig. 1**). She used decimal values for the probabilities:

$$P(\textit{No Triangle}) = 0.5$$
$$P(\textit{Obtuse}) = 0.29$$
$$P(\textit{Acute}) = 0.21$$
$$P(\textit{Right}) = 0$$

Katie's direct use of the areas in **figure 1** as probabilities led to a short discussion about how the "correct" answer to this question depended on interpretation of the question—that is, does the probability need to account for nontriangles?

Finally, Claudine presented her diagram and shared her answers:

Area of obtuse region = 0.39
Area of acute region = 0.22

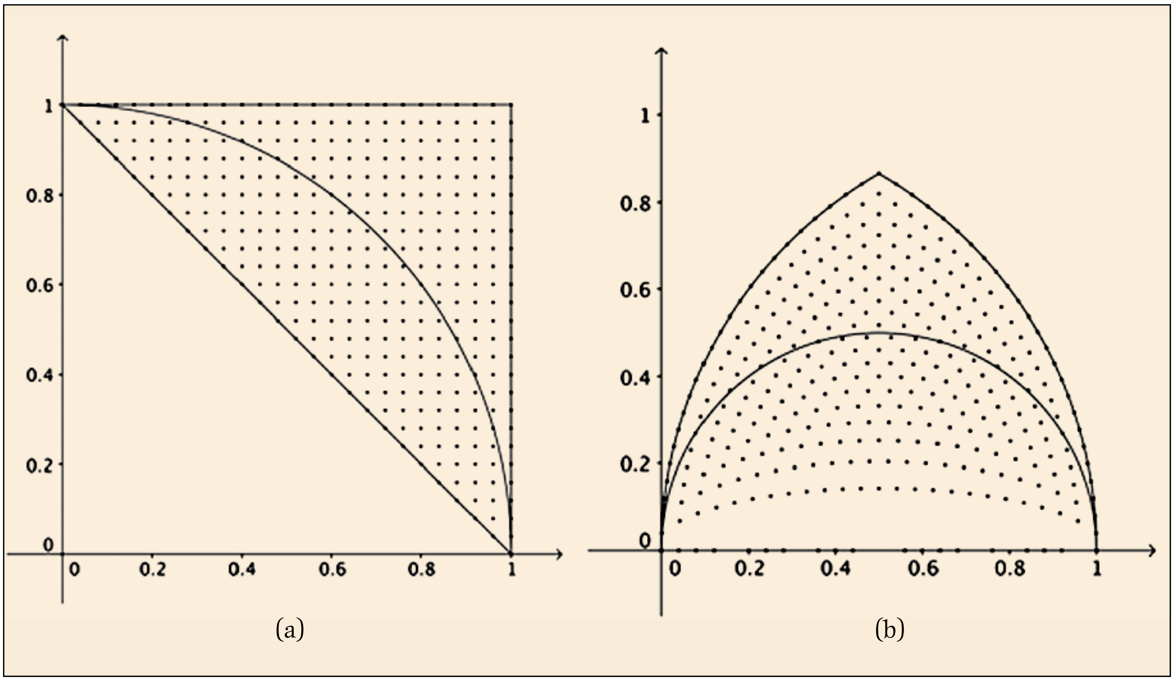

Fig. 7 The graphic generated by Oliver shows the consistent spacing of points *A* in Katie's solution and the variable spacing of points *E* in Claudine's solution.

At this point, things got interesting. I had not worked out Claudine's solution in advance, so I had not anticipated that Katie and Claudine would have different answers. After double-checking our calculations, one student suggested that there was some kind of stretching going on and that we had to compare the ratio of the areas in each solution. Even though the numbers differed, this student argued, the ratio would be constant. This approach sounded reasonable, but we found the following results:

0.285 : 0.215 = 1.326 : 1 (Katie's solution)

and

0.393 : 0.221 = 1.778 : 1 (Claudine's solution).

Further verifying that the ratios could not be the same, one student pointed out that Katie's areas were in terms of multiples of π alone, whereas Claudine's areas included multiples of π and $\sqrt{3}$. A simple scaling could never reconcile these solutions.

At this point, I was not sure what to do. This Random Triangle problem, which I thought I understood, had become a real problem. We put this problem in the "parking lot" of issues that we would return to later.

OBSERVATION AND COMPARISON

Following Erickson's (2001) method of using random variables to investigate perimeter-area relationships in rectangles, I created a dynamic simulation in GeoGebra to explore this problem. To construct Katie's solution, I generated random decimals between 0 and 1 for a and b and plotted $A = (a, b)$. The color of A was determined by a rule checking whether the triangle with side lengths a, b, and 1 was acute (green), obtuse (blue), or impossible (red).

To create Claudine's solution, I followed her geometric construction, starting with a segment from (0, 0) to (1, 0) in the second graphics window of the same GeoGebra file. Using the same values of a and b from the Katie's solution, I constructed a circle with radius a centered on the point (0, 0) and a circle with radius b centered on (1, 0). Point E was constructed to be the intersection of the two circles in the first quadrant and was connected to the points (0, 0) and (1, 0). The color of E was green if the vertex angle at E was acute or blue if it was obtuse. The simulation is available online at **www.nctm.org/mt066**.

Turning on the trace feature for point A in Katie's solution and point E in Claudine's solution created the expected pattern. Because the sketch was a simulation, GeoGebra could sample a few—or many—values for a and b, and the images from **figures 1** and **4** would emerge. **Figure 6** shows the simulations after 202 and 10,005 trials.

After 10,000 trials, I noted the lack of points along the segment from (0, 0) to (1, 0) in Claudine's solution. There was no corresponding gap between the random points in Katie's solution. I let the simulation run several times to convince myself that this gap in points was related to the model rather than to random variation. Something was going on, although at this point I was not sure whether I was seeing a rounding error or something deeper.

AN AHA! MOMENT

The day after I created the simulation, my Calculus for Teachers class was discussing applications of integrals. As my students worked on a set of problems, I mentioned the surprising conundrum that we had found in my Problem Solving course. I wondered aloud whether we could reconcile the solutions using integration because there had to be some kind of density function underlying Claudine's solution. This insight was inspired by an exercise in our textbook that required setting up and integrating a population density function to find the population of a circular city (Finney et al. 2006, p. 387, exercise 23). Of course, mentioning an unsolved problem to a group of mathematics teachers is sure to spark interest. They wanted to hear more, so I showed the students the problem and my GeoGebra simulation.

Coauthor Dent, a student in my calculus class, was so interested in the conundrum that he presented this problem and the simulation to his high school students at Boston University Academy. One of Nick's students, Oliver, created a graph to illustrate what was going on in Claudine's solution. He first set up and solved simultaneous equations for the intersection of two circles:

$$y = \sqrt{a^2 - x^2} \qquad (1)$$

$$y = \sqrt{b^2 - \left(x - 1\right)^2} \qquad (2)$$

Setting (1) and (2) equal and solving for x yields

$$x = (1/2)(1 + a^2 - b^2).$$

Substituting x back into (1) gives y in terms of a and b:

$$y = \sqrt{a^2 - \left(\frac{1}{2}\left(1 + a^2 - b^2\right)\right)^2}$$

Using these values for (x, y), Oliver generated a list of points in GeoGebra by letting a and b vary from 0 to 1 in increments of 0.05. Oliver's graph is reproduced in **figure 7b** with Claudine's image superimposed. For comparison, the same set of points (a, b) are also graphed on Katie's solution. The key feature revealed in **figure 7a** is that the distribution, or *density*, of points in Claudine's solution is not uniform: Point E is more likely to land in some regions than others in Claudine's diagram. This was not the case in Katie's diagram, in which all the points are uniformly distributed. The uneven distribution of points explained why Claudine's correct area calculations did not yield the correct probabilities. I brought this image to class, and we discussed how geometric probability problems rest on the assumption (often unstated) that points are evenly distributed.

PERSEVERANCE AND PROOF

As we revisited this problem in both my courses, Oliver's images helped us see how to reconcile the solutions. The spacing of points just above the x-axis in Claudine's solution shows that the gap that we observed in the simulation was not due to a rounding error. Instead, a very small range of values for a and b values result in triangles whose vertex, E, is located just above the segment for side c. Further, the pattern of points in Oliver's representation pointed the way to setting up a density function because it was reminiscent of Handa and Yakes's (2010) analysis of the distribution of random points on a dartboard.

Dent reasoned that if we could find a function to define the density of points in Claudine's solution, then we could integrate the density function to recover the probabilities. As with the traditional Dartboard problem, the density of points is a function of two variables, so this situation was more complicated than the one-variable probability densities that we were discussing in my calculus course. Nick happened to be enrolled in multivariable calculus, and he dove in to solve the problem (Nick's solution appears in the **appendix** at the end of this article).

While Dent wrestled with defining and integrating the density function, the mystery of the Random Triangle problem inspired several of my students to share this problem with friends and in their high school mathematics courses. For example, Danielle, who was student teaching in a high school geometry class, presented this problem along with the GeoGebra simulation to her students. Danielle's cooperating teacher doubted whether high school geometry students could tackle such a difficult problem, but she allowed Danielle to try it out. Much to the teacher's surprise, the students were able to arrive at the solution by following a process similar to what we did in my Problem Solving course. With some guidance from Danielle, they started by guessing a few values and then generalized by creating a graph like that shown in **figure 1**. Danielle's students felt very accomplished when she shared that *she* was working on this problem in her college mathematics class.

REAL PROBLEM SOLVING

This article has told the story of how one student's unique solution to a problem created new opportunities to explore connections among geometry, algebra, probability, and calculus, both single and multivariable. The dynamic simulation provided an inspiration and motivation for persevering with this problem and making sense of what was happening.

As a teacher, I noticed that once we had a real mystery to ponder, my students' interest in this problem grew dramatically. Students like Danielle shared this problem with their high school students and used the fact that she was still working on understanding Claudine's solution to spark discussion. Claudine and Nick both took this problem to their university-level mathematics classes. Ultimately, this relatively simple problem proved to be a rich source of mathematical exploration and growth for all of us, across the mathematical spectrum.

One of my goals for presenting this problem was to show my students the Common Core's SMP 1—"Make sense of problems and persevere in solving them"—in action. This problem succeeded beyond my expectations because Claudine's unexpected solution repositioned me: I no longer knew all the answers, and my students could see me as a problem solver. In fact, Claudine's solution pushed me to create the GeoGebra simulation and to explore this problem in far more depth.

Not knowing the answer can be an uncomfortable space for a teacher. However, a real problem is also an invaluable opportunity to model for students what problem solving—real problem solving—looks like. According to feedback from my students, the struggle and uncertainty paid off. Seeing me struggle with a problem alongside my students showed them that problem solving is possible—and powerful.

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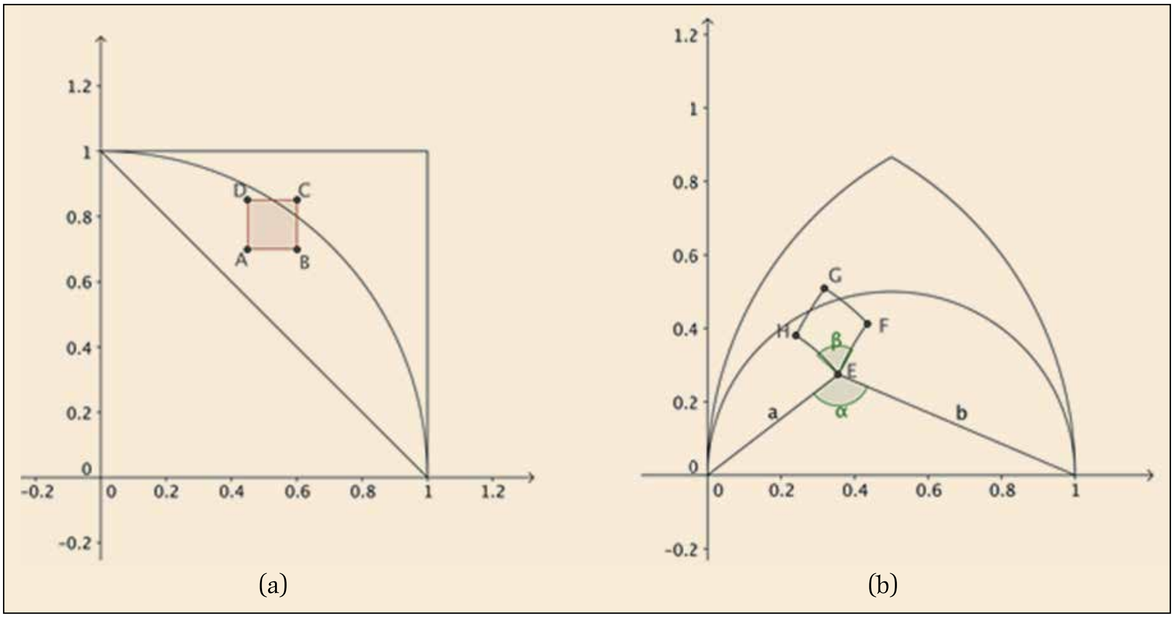

Fig. 8 The set of points inside square *ABCD* (**a**) corresponds to the region inside *EFGH* (**b**).

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APPENDIX: DENT'S SOLUTION

We reconciled the two solutions to the triangle probability problem using the principle that integrating a density function of a laminate yields the "mass" of the object (Larson, Hostetler, and Edwards 2006). In this case, the density function describes the density of points in Claudine's solution to the Random Triangle problem, and the integral recovers the probability of constructing a triangle (whether acute, obtuse, or right) from Katie's solution.

Step 1: Define a Density Function

Compare a square region in Katie's solution with the corresponding figure in Claudine's solution (see **fig. 8**).

In **figure 8a**, we let $A = (a, b)$ and define square *ABCD* with side length Δr built on *A*. The area of *ABCD* is $(\Delta r)^2$. **Figure 8b** shows region *EFGH* made of intercepting arcs built on concentric circles with a difference in radii of Δr. One pair of circles is $x^2 + y^2 = a^2$ and $x^2 + y^2 = (a + \Delta r)^2$. The second pair of circles is $(x - 1)^2 + y^2 = b^2$ and $(x - 1)^2 + y^2 = (b + \Delta r)^2$.

Every point in *ABCD* corresponds to a point in *EFGH*. Therefore, the probability of a random point landing in *EFGH* is the same as the probability of landing in *ABCD*.

In geometric probability, the probability density is defined as

$$\textit{Probability density} = \frac{\text{probability of landing in } ABCD}{\text{area of } ABCD}. \quad (1)$$

We know that the area of *ABCD* in **figure 8a** is $(\Delta r)^2$. Since points are uniformly distributed in Katie's solution (i.e., the probability density is uniformly equal to 1), the probability of landing in *ABCD* is also $(\Delta r)^2$.

By construction, we know that the probability of landing in region *EFGH* in **figure 8b** is the same as the probability of landing in *ABCD*, but the area is more difficult to calculate. However, since we will be integrating this density function, we can use an argument based on characteristics of *EFGH* as Δr becomes small.

As $\Delta r \to 0$, *EFGH* becomes very close to a rhombus with side length *s*, and the boundary arcs of *EFGH* can be approximated with segments (similar to the way curves are approximated as segments in the derivation of arc length in calculus). **Figures 9a** and **9b** show the region *EFGH* when $\Delta r = 0.2$ and $\Delta r = 0.01$.

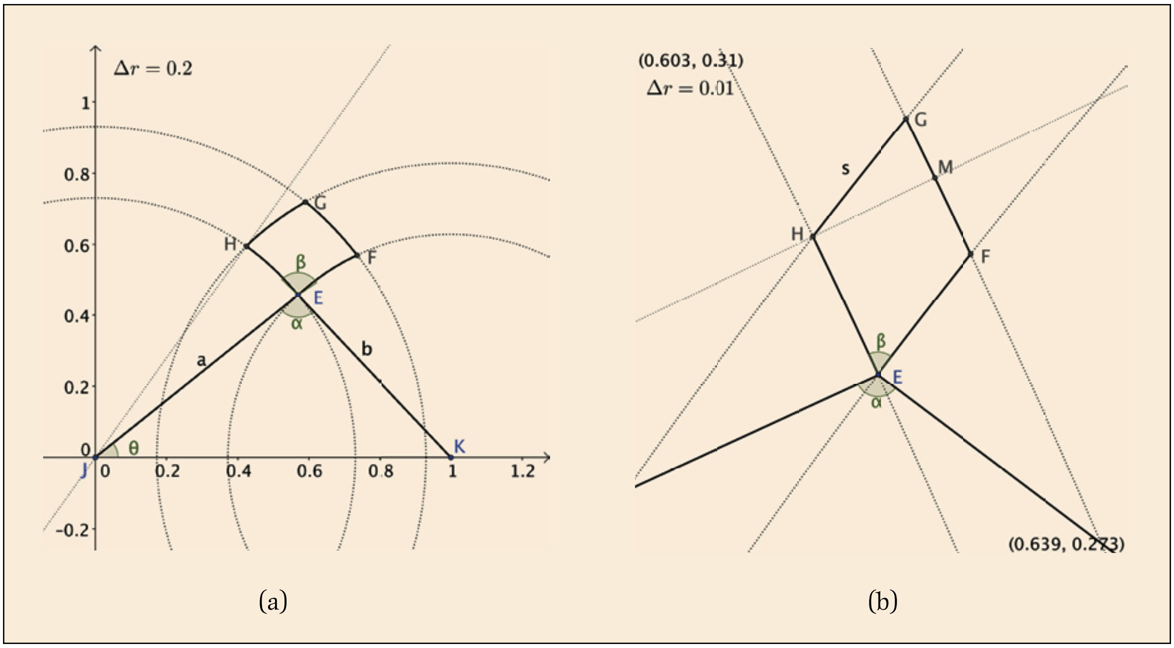

Fig. 9 As $\Delta r \to 0$, *EFGH* can be approximated by a rhombus.

For a small Δr, we have $\overline{EH} \perp \overline{JE}$ and $\overline{EF} \perp \overline{KE}$ because tangent segments are perpendicular to the radius at the point of tangency. Thus, $\alpha + \beta = 180°$. In **figure 9b**, the auxiliary line $\overline{HM}$ was constructed by extending the radius of the circle centered at $J = (0, 0)$ through H to M. By construction, we know that the length of $\overline{HM} = \Delta r$. We can see that $\sin(\beta) = \Delta r/s$, so $s = \Delta r/\sin(\beta)$.

Finally, using the area of a parallelogram, we find that

$$Area(EFGH) = s\Delta r = \frac{\Delta r}{\sin(\beta)}\Delta r = (\Delta r)^2/\sin(\beta).$$

Recall that $\alpha + \beta = 180°$, so by a trigonometric identity, we have $\sin(\beta) = \sin(\alpha)$. Making this substitution, we express the area in terms of $\sin(\alpha)$, which is convenient because α is part of the original triangle: $Area(EFGH) = (\Delta r)^2/\sin(\alpha)$. We have a new probability density, analogous to equation (1), for Claudine's solution:

$$\begin{aligned} Probability\ density &= \frac{\text{probability of landing in } EFGH}{\text{area of } EFGH} \\ &= \frac{(\Delta r)^2}{(\Delta r)^2/\sin(\alpha)} \\ &= \sin(\alpha) \end{aligned}$$

Step 2: Integrate and Verify

The final step in recovering the probability is to integrate the density function over the area to arrive at the probabilities of each region.

We have the density function in terms of α, but writing $\sin(\alpha)$ as a function of (a, b) in rectangular coordinates is cumbersome. However, since Claudine's construction is built on circles, it is relatively easy to rewrite $\sin(\alpha)$ as a polar function of r and θ. We have $r = a$ from our original problem and identify θ in **figure 9b**. By the law of sines, we have $\sin(\alpha)/1 = \sin(\theta)/b$. Also, using the law of cosines, we can substitute for b in terms of a and θ:

$$b^2 = 1^2 + a^2 - 2(1)(a)\cos(\theta)$$

So, finally, we now have

$$Probability\ density(a,\ \theta) = \frac{\sin(\theta)}{\sqrt{1 + a^2 - 2a\cos(\theta)}}. \quad (2)$$

From multivariable calculus, double integrals can be transformed between rectangular and polar coordinates using the substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$ along with the identity

$$\iint f(x, y)\,dy\,dx = \iint f(r, \theta)\,r\,d\theta\,dr$$

(Larson, Hostetler, and Edwards 2006). To evaluate the double integral, we must choose appropriate bounds on the integral. The radius is bounded by $0 \le r \le 1$, while the angle varies from 0 to the boundary formed by $(x - 1)^2 + y^2 = 1$, which is the polar curve $r = 2\cos(\theta)$ or, equivalently, $\theta = \cos^{-1}(r/2)$. (We do not need to worry about the domain of $\cos^{-1}(r/2)$ because our attention is restricted to the first quadrant.) Integrating the density function over the entire area with a computer algebra system we get

$$\int_0^1 \int_0^{\cos^{-1}(r/2)} \frac{\sin(\theta)}{\sqrt{r^2 - 2r\cos(\theta) + 1}}\, r\, d\theta\, dr = \frac{1}{2}. \qquad (3)$$

This shows that the probability associated with the whole region is 1/2, which we expected because the area of possible triangles in Katie's solution was 1/2. Restricting ourselves to the region of obtuse triangles will require adjusting the bounds on θ. The semicircle of right triangles in Claudine's representation is given by the equation $(x - 1/2)^2 + y^2 = (1/2)^2$, which in polar coordinates yields a new upper bound $r = \cos(\theta)$ or $\theta = \cos^{-1}(r)$. Again using a CAS to evaluate the double integral, we found that

$$\int_0^1 \int_0^{\cos^{-1}(r)} \frac{\sin(\theta)}{\sqrt{r^2 - 2r\cos(\theta) + 1}}\, r\, d\theta\, dr = \frac{\pi}{4} - \frac{1}{2}. \qquad (4)$$

Now we see that the probability of forming an obtuse triangle is $\pi/4 - 1/2$. Finally, subtracting the result (4) from (3) will give the probability of forming an acute triangle:

$$\frac{1}{2} - \left(\frac{\pi}{4} - \frac{1}{2}\right) = 1 - \frac{\pi}{4}.$$


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