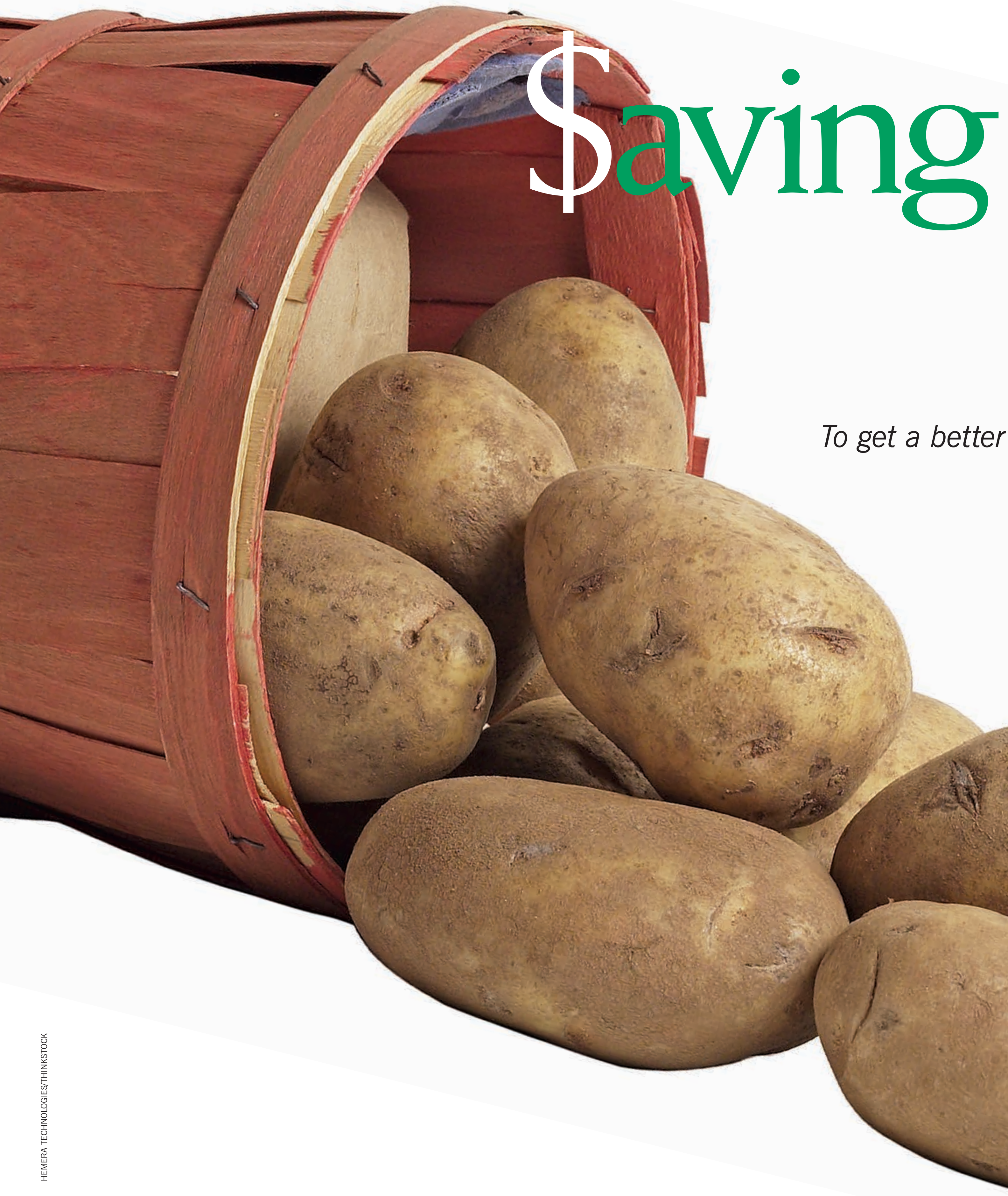

# $aving

*To get a better*

HEMERA TECHNOLOGIES/THINKSTOCK

# Money

# USING PROPORTIONAL REASONING

*buy, students employ various strategies to compare grocery prices.*

Jessica A. de la Cruz and Sandra Garney

How would your middle school students solve this missing value problem:

> If 4 pounds of potatoes cost $6.00, how much would 10 pounds of potatoes cost?

Would they be more likely to apply the cross-multiplication algorithm, as opposed to simpler multiplicative reasoning approaches? Although cross multiplication results in a correct answer, students using this method do not necessarily understand proportionality. Rather than the more commonly used missing-value problems, like the previous example, we suggest posing comparison problems to help students recognize the underlying multiplicative relationship that exists within a proportion.

Consider this comparison problem:

> Which is a better price for potatoes: $1.29 for 10 pounds or $4.99 for 20 pounds?

Here, a factor-of-change strategy is intuitive (if the number of pounds doubles, then so should the price), and it emphasizes the multiplicative relationship between the two ratios. Moreover, cross multiplication is likely to be unsuccessful in determining the better price because students must interpret the relationship between the cross products. **Figure 1** presents an example of cross multiplication when applied to compare ratios.

Many teachers would agree that once cross multiplication is introduced, their students tend to apply it by rote, abandoning all previously learned proportional reasoning strategies. Although cross multiplication is typically the most emphasized strategy in textbooks for solving missing-value proportion problems, many researchers believe that an overemphasis of this strategy is the root of students' difficulties with proportional reasoning. One study even found that students who were taught the cross multiplication strategy were actually

**Fig. 1** Using a cross-multiplication strategy to compare two prices yields two products that are difficult to interpret in terms of the original scenario.

$$\frac{\$2.50}{2\text{ lb.}} \stackrel{?}{=} \frac{\$3.25}{3\text{ lb.}}$$

$$(\$2.50)\times(3\text{ lb.}) \stackrel{?}{=} (\$3.25)\times(2\text{ lb.})$$

**Fig. 2** The comparison task and anticipated strategies for an integer factor of change (× 2) problem are illustrated.

1. Which is the better deal for potatoes?


A


B

**Factor of Change (× 2)**

$$A: \frac{\$1.29}{10\text{ lb.}} \overset{\times 2}{=} \frac{\$2.58}{20\text{ lb.}} \qquad B: \frac{\$4.99}{20\text{ lb.}}$$

**Unit Rate**

$$A: \frac{\$1.29}{10\text{ lb.}} \overset{\div 10}{=} \frac{\$0.13}{1\text{ lb.}} \qquad B: \frac{\$4.99}{20\text{ lb.}} \overset{\div 20}{=} \frac{\$0.25}{1\text{ lb.}}$$

less successful when solving proportion problems than students who were never taught the algorithm (Fleener, Westbrook, and Rogers 1993). Additionally, repetitive application of cross multiplication without knowledge of other proportional reasoning strategies is not enough to be considered proportional reasoning (Cramer, Post, and Currier, 1993; Fleener, Westbrook, and Rogers 1993). Students should fully develop more intuitive strategies, such as factor of change or unit rate strategies, before being introduced to cross multiplication. These intuitive strategies help students better understand the multiplicative relationship between proportional ratios. However, many textbooks heavily emphasize cross multiplication and leave a gap where teachers must develop other ways to foster the creation and use of different proportional reasoning strategies.

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It is beneficial for students to discover intuitive strategies, as opposed to the teacher presenting strategies to them. Certain proportional reasoning tasks are more likely to elicit intuitive strategies than other tasks. The strategies that students are apt to use when approaching a task, as well as the likelihood of a student's success or failure solving it, are influenced by that task's context and numerical structure (de la Cruz 2013). Thus, teachers can encourage the development of particular strategies by carefully selecting the tasks that students will complete. Furthermore, implementing the Five Practices (Smith et al. 2009) can assist teachers in structuring the whole-class sharing of student-generated strategies in an organized and purposeful way. Considering the effects that task characteristics can have on strategy choices, we designed the Better Buy Lesson, which we describe here.

## THE FIVE PRACTICES MODEL

Smith and colleagues (2009) present a model to support and prepare teachers to incorporate students' thinking into classroom discussion. Focusing on the Five Practices helps teachers by limiting the in-the-moment decisions that are sometimes frightening aspects of

**Fig. 3** The comparison task and anticipated strategies for an integer factor of change (× 10) problem, with the unit rate provided, are explored next.

2. Which is the better deal for potatoes?


A


B

**Unit Rate**

$$\text{A: } \frac{\$1.29}{10 \text{ lb.}} \overset{\div 10}{=} \frac{\$0.13}{1 \text{ lb.}} \qquad \text{B: } \frac{\$0.69}{1 \text{ lb.}}$$

**Factor of Change (× 10)**

$$\text{A: } \frac{\$1.29}{10 \text{ lb.}} \qquad \text{B: } \frac{\$0.69}{1 \text{ lb.}} \overset{\times 10}{=} \frac{\$6.90}{10 \text{ lb.}}$$

student-centered teaching. Moreover, it better prepares teachers to highlight the facets of students' thinking that tie specifically to the instructional goal. The Five Practices include the following: (1) anticipating, (2) monitoring, (3) selecting, (4) sequencing, and (5) connecting. When planning the Better Buy Lesson, we chose challenging mathematical tasks while anticipating the strategies that students would use when solving. Next, we selected the strategies we aimed to share in the discussion portion of the lesson by considering our ultimate instructional goals. Then we predicted how we would sequence the shared strategies, with the understanding that this sequence may be adapted, depending on what we observed when monitoring the classwork. Finally, we planned how we would connect the shared strategies to each other and to our instructional goals.

## THE BETTER BUY ACTIVITY

Students, working in pairs, were asked to determine the better deal when given two different prices and quantities for similar items found in competing grocery store ads. They were instructed to use any strategy that they could fully explain to the class. In total, there were four comparison tasks (see **figs. 2–5**). Before the lesson, we chose each task carefully, anticipating strategies we thought students would use and after analyzing each task's numerical structure.

### *Choosing the Comparison Tasks and Anticipating Strategies*

First, we predicted that students would have little success applying cross multiplication to compare the ratios, which is consistent with Singh's (2000) research. When the rates being compared are not proportionally related, interpreting the cross products is difficult (see **fig. 1**). It is clear from the unequal cross products that the ratios are not equivalent; however, it is not clear which one is the better buy. This meant that students would likely employ alternative strategies.

Second, we had two goals in mind when analyzing and selecting the four comparisons: To encourage flexible use of several proportional reasoning strategies and to emphasize the multiplicative nature of proportional ratios. Depending on the strategy, we chose particular numerical structures known to influence different problem-solving approaches (Tjoe and de la Torre 2013)

According to Lesh, Behr, and Post (1987), the presence of an integer factor of change between the ratios increases the likelihood that students would apply a factor of change strategy, also referred to in the literature as a building up through multiplication strategy (Steinthorsdottir and Sriraman 2009). The following comparison would likely encourage the use of a factor of change strategy: \$15.00 for 4 pounds of dog food at store A versus \$78.00 for 24 pounds at store B. At store A, we can determine the price for 24 pounds using a factor of change of 6:

$$\frac{\$15.00}{4 \text{ lb.}} = \frac{\$90.00}{24 \text{ lb.}}$$

The presence of an integer factor of change within one of the ratios (i.e., an integer unit rate), coupled with the absence of an integer factor of change between the ratios, encourages students to apply unit rate strategies. For instance, \$15.00 for 5 pounds of dog food at store A versus $76.00 for 19 pounds would likely be solved using a unit rate strategy: At store A,

$$\frac{\$15.00}{5\text{ lb.}} = \frac{\$3.00}{1\text{ lb.}};$$

at store B,

$$\frac{\$76.00}{19\text{ lb.}} = \frac{\$4.00}{1\text{ lb.}}.$$

**Figures 2–5** present the four comparison tasks we created, highlight the numerical structure for each, and list the approaches that we anticipated students would use. When designing the activity, we aimed to have students perform these strategies: factor of change, unit rate, common denominator, and combination strategies. We looked for these specific strategies when we monitored the activity.

***Monitoring Students' Work***

According to Smith et al. (2009), teachers should monitor their students' thinking and strategies as they work to productively determine who should share and what should be shared in class discussion. Without careful monitoring and selecting, the discussion can turn into a "show and tell" of disconnected strategies and may not deepen students' understandings. **Figure 6** depicts the table we used to record our assessments throughout the monitoring process. It also indicates decisions that resulted when we anticipated students' approaches while also considering our instructional goal. We included an additional row at the bottom of the table to capture any unforeseen strategies as well as note incorrect additive approaches.

**Fig. 4** This comparison task and anticipated strategies show a problem with no integer factor of change.

3. Which is the better deal for 12 packs of Coca-Cola?


A


B

**Unit Rate**

$$\text{A: } \frac{\$9.00}{4\text{ packs}} \xrightarrow{\div 4} \frac{\$2.25}{1\text{ pack}} \qquad \text{B: } \frac{\$22.50}{10\text{ packs}} \xrightarrow{\div 10} \frac{\$2.25}{1\text{ pack}}$$

**Common Denominator**

$$\text{A: } \frac{\$9.00}{4\text{ packs}} \xrightarrow{\times 5} \frac{\$45.00}{20\text{ packs}} \qquad \text{B: } \frac{\$22.50}{10\text{ packs}} \xrightarrow{\times 2} \frac{\$45.00}{20\text{ packs}}$$

**Combination of Buildup and Reduction**

$$\text{A: } \frac{\$9.00}{4\text{ packs}} \xrightarrow{\times 2} \frac{\$18.00}{8\text{ packs}} \qquad \text{B: } \frac{\$22.50}{10\text{ packs}}$$

$$\frac{\$9.00}{4\text{ packs}} \xrightarrow{\div 2} \frac{\$4.50}{2\text{ packs}}$$

$$\frac{\$9.00}{4\text{ packs}} = \frac{\$18 + \$4.50}{8 + 2\text{ packs}} = \frac{\$22.50}{10\text{ packs}}$$

***Selecting, Sequencing, and Connecting Students' Work***

After monitoring the students' work on the four comparison tasks and referencing our monitoring tool, specific groups were selected to share their strategies with the class. A pair who used long division to calculate the unit prices per pound of potatoes, in problem number 1, was asked to share first. Their work is depicted in **figure 7a**. Next, a pair was chosen to share their

**Fig. 5** The task and anticipated strategies for a problem involved comparing three ratios.

4. Which is the better deal for paper towels?


**Unit Rate**

÷ 8

A: $\dfrac{\$8.99}{8 \text{ rolls}} = \dfrac{\$1.12}{1 \text{ roll}}$

÷ 8

÷ 6

B: $\dfrac{\$7.99}{6 \text{ rolls}} = \dfrac{\$1.33}{1 \text{ roll}}$

÷ 6

÷ 2

C: $\dfrac{\$3.00}{2 \text{ rolls}} = \dfrac{\$1.50}{1 \text{ roll}}$

÷ 2

**Reduction**

÷ 4

A: = $\dfrac{\$8.99}{8 \text{ rolls}} = \dfrac{\$2.25}{2 \text{ rolls}}$

÷ 4

÷ 3

B: $\dfrac{\$7.99}{6 \text{ rolls}} = \dfrac{\$2.66}{2 \text{ rolls}}$

÷ 3

C: $\dfrac{\$3.00}{2 \text{ rolls}}$

**Common Denominator**

× 3

A: = $\dfrac{\$8.99}{8 \text{ rolls}} = \dfrac{\$26.97}{24 \text{ rolls}}$

× 3

× 4

B: = $\dfrac{\$7.99}{6 \text{ rolls}} = \dfrac{\$31.96}{24 \text{ rolls}}$

× 4

× 12

C: $\dfrac{\$3.00}{2 \text{ rolls}} = \dfrac{\$36.00}{24 \text{ rolls}}$

× 12

factor-of-change strategy (see **fig. 7b**). This strategy was presented after the unit rate strategy to illustrate the simplicity of the computations involved, in contrast to the previous method. Thus, the first comparison task led to a discussion of student-generated unit rate and factor of change strategies and motivated students to consider when one strategy would be more easily applied than another. Additionally, the teacher seized the opportunity to point out that a multiplicative relationship between ratios, as shown in the factor of change strategy, always exists when ratios are proportional. Further, the class discussed how the unit rate strategy is similar to a factor of change strategy. **Figure 8** illustrates how we find the unit price for potatoes at store A by multiplying the provided ratio by a factor of one-tenth, or divide by ten, to get a unit in the denominator.

If someone in our class had used an additive approach to compare these ratios, we would have addressed it by connecting to the context. For instance, if someone had explained that they added 10 pounds to get from 10 pounds to 20 pounds, so they also added \$10.00 to the cost to get \$11.29, we would have directed the class to notice that this would mean that the first 10 pounds cost \$1.29, but the second 10 pounds cost \$10.00. Since the cost for the same weight of potatoes should be the same, this additive strategy does not make sense.

Task 2 also involves potatoes; however, in this task one of the provided prices was given as a unit rate. The inclusion of a unit rate further encouraged the use of a unit rate strategy. This task was incorporated to ensure that a unit rate strategy would be

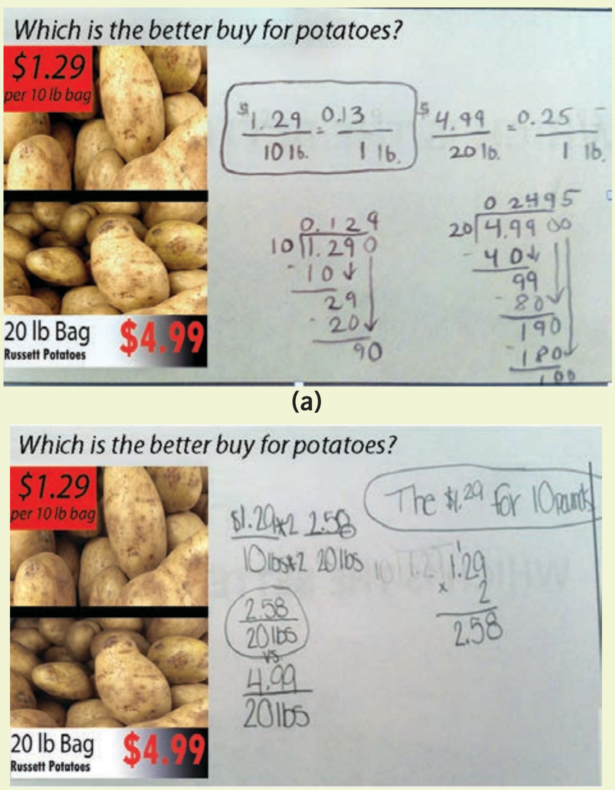
**Fig. 6** This monitoring tool helped us select and sequence who and what would be shared in the whole-class discussion.

| | Strategy | Who and What | Order |
|---|---|---|---|
| Task 1 | Unit Rate | | First |
| | Factor of Change | | Second |
| Task 2 | Unit Rate | | TBD |
| | Factor of Change | | TBD |
| Task 3 | Unit Rate | | First (or omit) |
| | Common Denominator | | Second |
| | Combination | | Third |
| Task 4 | Unit Rate | | First (or omit) |
| | Common Denominator | | Second |
| | Reduction | | Third |
| Task ___ | Other | | |

*Source:* Adapted from Smith et al. (2009)

Note: The cells that are completed in the "order" column specify our anticipated sequence prior to monitoring the classwork. TBD indicates that the order was determined during the monitoring process based on the frequency of that strategy's use, with the most common strategy shared first.

**Fig. 7** One pair of students used the unit rate strategy on task 1 (a); another used the factor of change strategy for the same task (b).




(a)

(b)

shared. Unlike task 1, we determined the order in which the strategies would be presented while monitoring the classwork. To provide validation, we began with the most commonly used strategy. Again, the class discussed how both strategies, factor of change and unit rate, were related by looking at the multiplicative change involved in each.

In task 3, students compared the prices for differing numbers of 12 packs of soft drinks, $9.00 for 4 at store A versus $22.50 for 10 at store B. This numerical structure is unique from the previous two tasks in that it involves equivalent ratios and the factor of change between ratios is not an integer. The aim of this task was to elicit a common denominator strategy (e.g., find the cost of 20 or 40 of the 12 packs at each store) and a combination strategy (e.g., find the cost of 10 of the 12 packs at each store, by finding the cost of 8 and 2 of the 12 packs at store A and combining). **Figure 9** portrays the combination strategy that one group shared.

In the class discussion, we selected two groups who had used the two intended strategies to present their processes for the class. We sequenced the strategies in order of sophistication, with the common denominator strategy presented first. This strategy was deemed less sophisticated because it connected to the students' prior knowledge regarding fraction equivalence. According to Smith et al. (2009), it can be beneficial to begin with a strategy that is more familiar to students to validate their thinking and allow for connections between prior knowledge (equivalent fractions) and new knowledge (equivalent ratios). The class then discussed the similarities and differences between the common denominator strategy just witnessed and the factor of change strategies seen for task 1 and 2. Together we recognized that the

**Fig. 8** The teacher connected the factor of change strategy for finding an equivalent ratio to the unit rate strategy for finding an equivalent unit rate by illustrating that the unit rate strategy involves multiplying by a fractional factor of change.

Factor of Change

$$\frac{\$1.29}{10} \xrightarrow{\times 2} \frac{\$2.58}{20}$$

$$\frac{\$1.29}{10} = \frac{\$2.58}{20}$$

Unit Rate

$$\frac{\$1.29}{10} \xrightarrow{\times \frac{1}{10}} \frac{\$0.13}{1}$$

$$\frac{\$1.29}{10} = \frac{\$0.13}{1}$$

**Fig. 9** A combination strategy involving factor of change and reduction strategies was shared by one group of students.

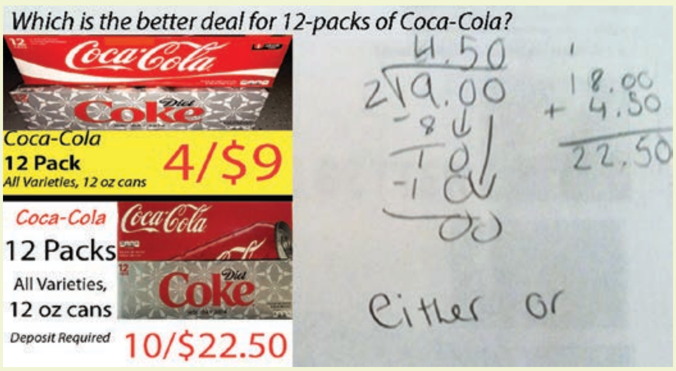



**Fig. 10** The teacher elaborated on the student-generated strategy shared in **figure 5**.

Step 1:
Factor of Change Strategy:

$$\frac{\$9.00}{4 \text{ lb.}} \xrightarrow{\times 2} \frac{x}{8 \text{ lb.}}$$

$$\frac{\$9.00}{4 \text{ lb.}} = \frac{x}{8 \text{ lb.}}, x = \$18.00 \Rightarrow \$18.00 \text{ for } 8 \text{ lb.}$$

Step 2:
Reduction Strategy:

$$\frac{\$9.00}{4 \text{ lb.}} \xrightarrow{\div 2} \frac{y}{2 \text{ lb.}}$$

$$\frac{\$9.00}{4 \text{ lb.}} = \frac{y}{2 \text{ lb.}}, y = \$4.50 \Rightarrow \$4.50 \text{ for } 2 \text{ lb.}$$

Step 3: Combine:

$$8 \text{ lb.} + 2 \text{ lb.} = 10 \text{ lb.}$$

$$\$18.00 + \$4.50 = \$22.50 \Rightarrow \$22.50 \text{ for } 10 \text{ lb.}$$

common denominator strategy is a factor of change strategy where the common denominator found is not equal to either of the original denominators.

Second, we asked a group of students to explain their combination strategy, which was discovered by few students in the class. We then explicitly connected the students' work to the factor of change and reduction strategies discussed earlier by labeling each step according to the strategy it matched and labeling the multiplicative relationships with arrow diagrams, as shown in **figure 10**. We represented this combination strategy again, but more concretely, in a table (see **fig. 11**). Within the table, we used arrows to mark the multiplicative relationships. We chose to label the reduction from 4 to 2 as division by 2, as opposed to multiplication by 1/2, because our students are more comfortable operating with whole numbers; however, we asked the class what our factor of change would be if we were to think of it as multiplication instead of division, to reiterate that a factor of change always exists.

The final task asked students to compare three different deals for paper towels: 8 rolls for $8.99, 6 rolls for $7.99, and 2 rolls for $3.00. This task appeared last because there were three ratios to compare. The majority of our students used a unit rate strategy to compare the three ratios and, hence, the unit rate strategy was shared first. Next, when monitoring the groups as they worked, we noticed a reduction strategy, determining the price for 2 rolls according to each deal, and a common denominator strategy, finding the price for 24 rolls using each deal. Those groups were asked to detail their approaches for the class. We again connected these approaches to the ones shared earlier in the discussion by depicting, with arrows, the factor of change for each.

**Fig. 11** This model was drawn on the board to further clarify the combination strategy shared in **figure 10** and to illustrate the connection to the previously presented strategies.

| Number of 12 Packs | Cost ($) for 4 12-packs for $9.00 |
|---|---|
| 2 | 4.50 |
| 4 | 9.00 |
| 8 | 18.00 |
| 10 (2 + 8 = 10) | 22.50 (4.50 + 18.00 = 22.50) |

We also asked, "Could we have used a factor of change strategy to find the price for 10 rolls?" Students then realized that the factor of change was 2.5, which we related to the combination strategy in which we found the price for 2 groups of 4 rolls and for 1/2 group of 4 rolls. We reiterated that the strategies that we discussed (factor of change, unit rate, common denominator, and combination) were all related to the factor of change strategy because equal ratios always have a multiplicative relationship that can be represented with an arrow diagram. Using the Five Practices and our well-thought-out tasks enabled us to effectively facilitate this student-centered lesson while achieving our content goals.

## THE END RESULT: QUANTITATIVE REASONING

The Better Buy lesson not only provided an interesting and real-life context for studying proportional reasoning strategies but also required students to reason quantitatively and model with mathematics, two of the mathematical practices delineated within the Common Core State Standards for Mathematics (CCSSI 2010, pp. 6–8). Although this activity was used with an eighth-grade class to review and highlight the multiplicative structure of proportional situations, it is best suited for sixth-grade and seventh-grade audiences before cross multiplication and other proportional reasoning strategies are formally introduced. The students were so engaged in this activity that many groups finished the four assigned tasks and continued on to complete other grocery price comparisons.

Using the Five Practices model during the planning and implementation of this lesson in the classroom, we were able to effectively highlight multiple proportional reasoning strategies and their multiplicative properties while maintaining the student-centered aspect of our instruction. Allowing the students to generate their own methods for comparing the ratios based on their prior knowledge and intuitions enabled us to connect the formal ideas to their informal ones and, in turn, will lead to deeper understandings (de la Torre et al. 2013) of proportionality.

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