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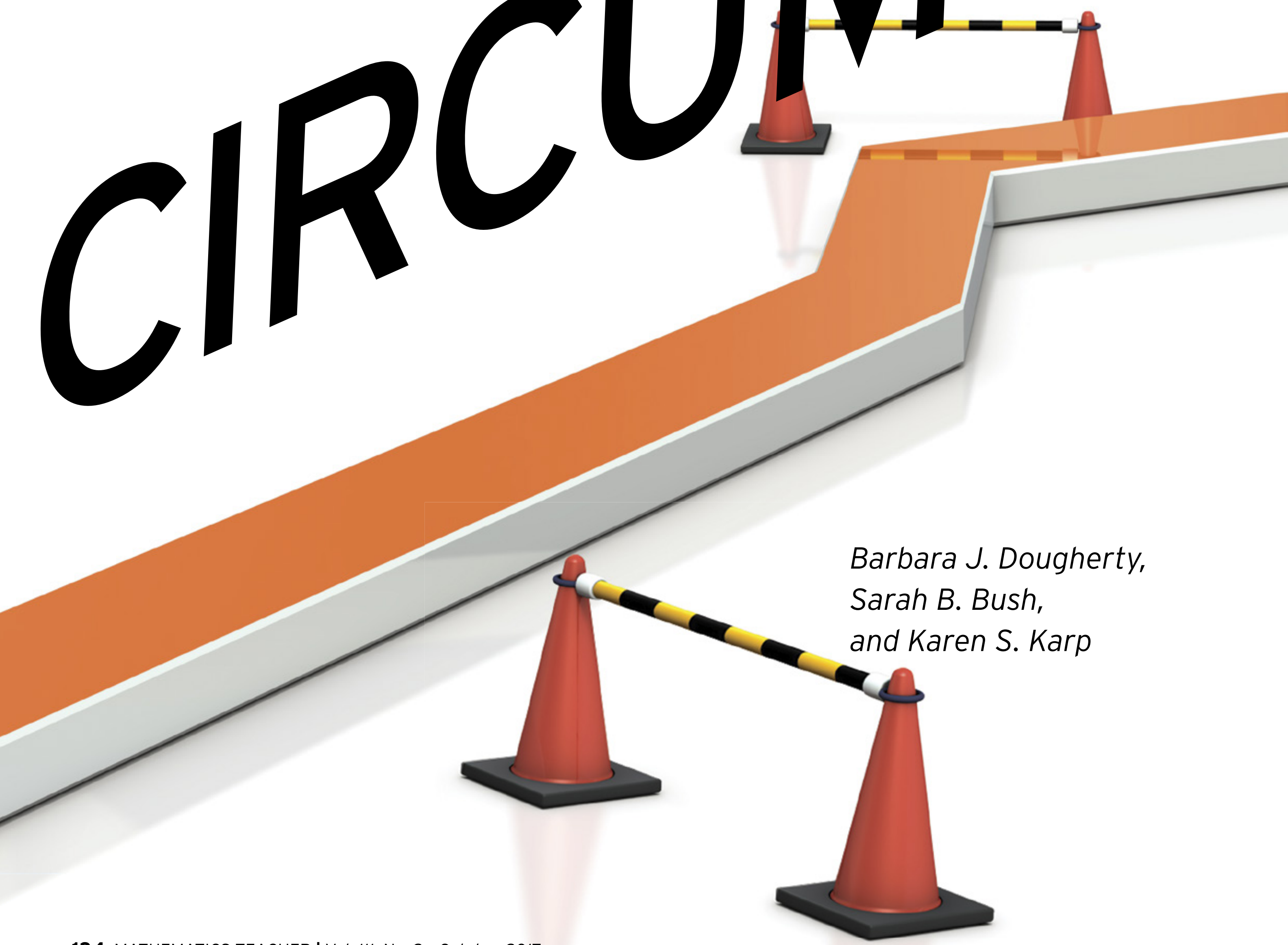

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NTING HIGH SCHOOL RULES THAT EXPIRE

CYCLONEPROJECT/THINKSTOCK

Consider the following prompt. How do you think your algebra students would respond if given the four choices below?

Mari said, "$2t$ is always greater than $t + 2$." Do you agree with Mari?

A. Yes, because multiplication always gives you a larger answer than addition.
B. Yes, because t is a positive number.
C. No, because multiplication is not the inverse of addition.
D. No, because it is possible that $2t$ can be equal to or less than $t + 2$.

In a recent study (Dougherty and Foegen 2014), 312 (41.6 percent) of 750 second-semester algebra 1 students selected *A* as their answer. In another example, a group of high school students responded to the following problem:

Dan challenged Amy to write an equation that has a solution of 3. Which equation could Amy have written?

A. $4 - x = 10 - 3x$
B. $3 + x = -(x + 3)$
C. $-2x = 6$
D. $x + 2 = 3$

Of 490 students who responded to this item, 28.8 percent indicated that the correct equation would be *D* because, as they explained in a focus group session, the answer always comes after the equal sign (Dougherty and Foegen 2014). (In fact, 72.5 percent of the responding students selected an incorrect answer.) We believe that all secondary mathematics teachers would agree that this is troublesome. The perpetuation of rules that expire (Karp, Bush, and Dougherty 2014; 2015), or rules that are taught in previous grades that no longer hold true, suggests that many secondary students harbor misconceptions from their elementary and middle-grades mathematics experiences as they progress to mathematics classes that are more sophisticated. High school students, particularly students who are struggling in mathematics class, often try to use these familiar but short-lived "rules" as strategies when attempting increasingly challenging tasks.

Here, we discuss fifteen rules that expire to highlight how these "rules"—which we have found prevalent in our many years working in schools, on websites, or in some cases, have taught and later regretted—can cause long-term problems. Many times what is taught in one grade can "expire" when students face topics and situations that are more sophisticated in the grades that follow. Additionally, the Common Core's (CCSSI 2010) Standards for Mathematical Practice (SMP) encourage precision, including the appropriate use of mathematics vocabulary. The purpose of this article is to describe some rules and terminology that can present difficulties over time in the mathematics classroom—ultimately hindering students' abilities to grasp more sophisticated mathematical ideas.

In the section below, we emphasize rules that are sometimes used with high school students, or at other times taught in middle grades, that initially appear to be useful but later cause complications. Although originally presented as well-intentioned shortcuts or mnemonics, later, when more advanced content or a higher demand for more complex understandings are required, students find that their rules backfire. Such experiences can be discouraging and can foster the idea that mathematics is a set of tricks and tips to memorize rather than a network of integrated concepts. For each rule that expires, we do the following (similar to Karp, Bush, and Dougherty 2014; 2015):

1. State the rule.
2. Discuss how students overgeneralize it.
3. Provide counterexamples.
4. State the point at which the rule begins to fall apart.

15 RULES THAT EXPIRE IN SECONDARY CLASSROOMS

1. –x Is Negative x.

Reading the expression $-x$ as *negative x* creates much confusion for students, especially those who believe that variables can assume only positive values. For these students, then, $-x$ means that the value of this expression will always be negative. This is problematic for students when x represents a negative number. If students consistently read the expression as *the opposite of x*, it would help them make sense of the expression. Expiration date: Grade 6 (6.NS.6.a)

2. The Absolute Value of a Number Is Just the Number.

Even in high school, students sometimes maintain the belief that the absolute value of a number is that same number with a positive sign. For example, $|-x| = x$ because the negative sign is simply dropped. Confusion often occurs when students carry this misunderstanding to algebraic contexts because they are unsure how it might be possible that $-|-x| = x$. Without understanding that the true meaning of an absolute value is its distance from zero on a number line, students may continue to have trouble interpreting absolute value scenarios that are more complex. Expiration date: Grade 6 (6.NS.7)

3. The Solution to an Equation Should Always Be Written So That the Variable Is on the Left of the Equal Sign: x = _______.

When students are frequently encouraged in an algebraic equation to "move" the variable to the left of the equal sign and position the answer (e.g., the constant) to the right of the equal sign, this "rule" confuses the meaning of the equal sign (Dougherty and Foegen 2011). This directive or "rule" is not mathematically necessary, because the equal sign indicates that two quantities are equivalent and when considering that relationship, the variables, operations, or constants can be located on either or both sides of the equal sign. Instead, students should realize that there are multiple ways to write equivalent equations.
Expiration date: Grade 6 (6.EE.4)

4. To Check if Two Expressions Are Equivalent, Substitute "1" for the Variable Because That Is the Easiest.

Students are sometimes taught to routinely substitute an "easy" number for the variable. Often, that number is 0, 1, or 2. If the substitution method is used to determine equivalent expressions, multiple values should be used to avoid coincidental situations. However, the substitution method does not always lead to a correct decision regarding equivalence, because it depends on the values students choose. For example, if the expressions are x^3 and x^2, and a student selects 0 and 1 to test, the resulting values could erroneously indicate that these expressions are equivalent.
Expiration date: Grade 6 (6.EE.4)

5. Use Cross-Simplifying with Multiplication of Fractions.

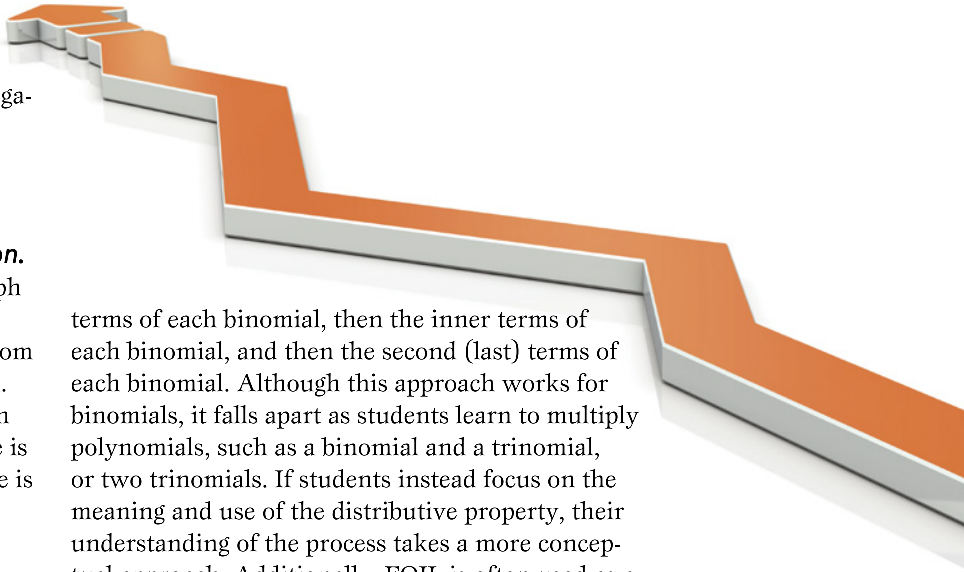
When multiplying rational expressions, students often confuse the use of cross multiplication (a method sometimes introduced for solving proportional equations) with the process for finding a product of fractions because they disregard the sign between the fractions and simply apply the process to juxtaposed rational expressions. They "cross" diagonally and recombine the numerators and denominators. In the problem

$$\frac{3}{n} \bullet \frac{8n}{9},$$

they would get an inaccurate product with a numerator of $27n$ and a denominator of $32n$.
Expiration date: Grade 7 (7.RP.2)

6. In the Expression 3 – 2x, the 2x Is Negative Because of the – Sign.

This rule often causes confusion among students in determining the difference between a direction and an operation sign, particularly when confronted with an expression such as $(7)(-3x)$. They are unclear about whether the – sign indicates direction or subtraction. Or they view $3 - 2x$ as $3(-2x)$.



This misconception is compounded when the negative (–) sign is read as *subtract* regardless of its meaning.
Expiration date: Grade 7 (7.NS.1.c)

7. If You Have a Straight Line, It Is a Function.

The visual representation of a function on a graph is compelling for students—and rightfully so—because a host of information can be garnered from the graph, such as the intercept, slope, and so on. However, relying on the graphical representation also leads students to think that any straight line is a function. This is not true if the slope of the line is undefined.
Expiration date: Grade 8 (8.EE.6)

8. An X|Y *Table Must Have a Set Number of Ordered Pairs.*

When students start creating $x|y$ tables, they may be given a parameter for the number of entries to include, such as, "Your table must have three (or some other set number of) ordered pairs." This may be done to ensure that they see an accurate shape of the graph, but it may lead them to the inflexible thinking that a graph contains exactly three points or that the shape of a graph is always revealed in three points.
Expiration date: Grade 8 (8.F.1)

9. You Cannot Find the Square Root of a Negative Number.

A rule that is often used for working with square roots is to look at the sign of the radicand first to determine if finding a square root is possible. Students are told that if the radicand is negative, it is impossible to find the square root. Additionally, students believe that square roots can only be positive values. These misconceptions lead to difficulties in dealing with imaginary numbers in future mathematics courses.
Expiration date: High School (N.CN.4)

10. The Square Root of a *Times the Square Root of* b *Always Equals the Square Root of* ab.

Although this might be true when initially teaching real numbers, this would not always be true when complex numbers are used. For example, the square root of –1 times the square root of –1 is i^2 which is not equal to 1 but to –1.
Expiration date: High School (N.CN.4)

11. When Multiplying, Use FOIL—First, Outer, Inner, Last.

Students may be taught to FOIL, that is, to multiply the first term in the first binomial by the first term in the second binomial, then multiply the outer terms of each binomial, then the inner terms of each binomial, and then the second (last) terms of each binomial. Although this approach works for binomials, it falls apart as students learn to multiply polynomials, such as a binomial and a trinomial, or two trinomials. If students instead focus on the meaning and use of the distributive property, their understanding of the process takes a more conceptual approach. Additionally, FOIL is often used as a verb, such as, "I just FOILED it." FOIL represents an acronym, not an action.
Expiration date: High School (A.APR.1)

12. To Solve an Equation, Move Numbers and Letters across the Magic River.

Students are often told that when solving an equation, such as $3x - 6 = 2x + 4$, they should move the 6 over to the other side of the equal sign. To help students understand the "moving" process, the equal sign is represented as two vertical and wavy lines representing a river (see **fig. 1**). Terms of the equation are moved back and forth across the *river* until the variable and constant are isolated. The focus on moving the terms takes away from the conceptual understanding of the relational properties of equality. Expiration date: High School (A.REI.1)

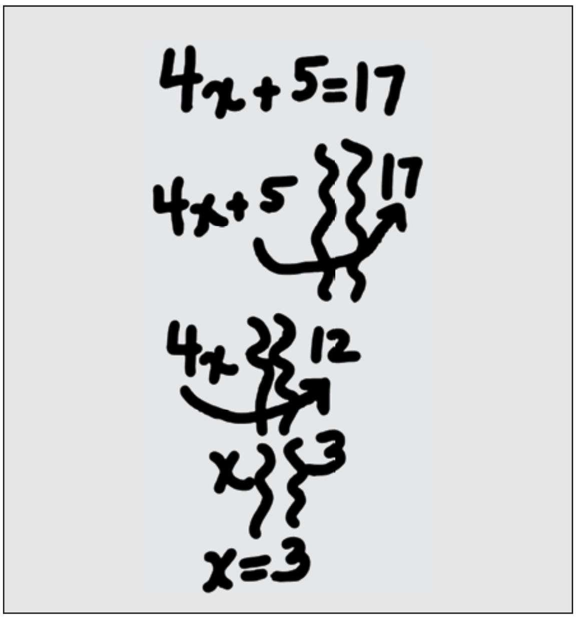


Fig. 1 The river metaphor for an equals sign promotes the idea that mathematics is something magical, without rhyme or reason.

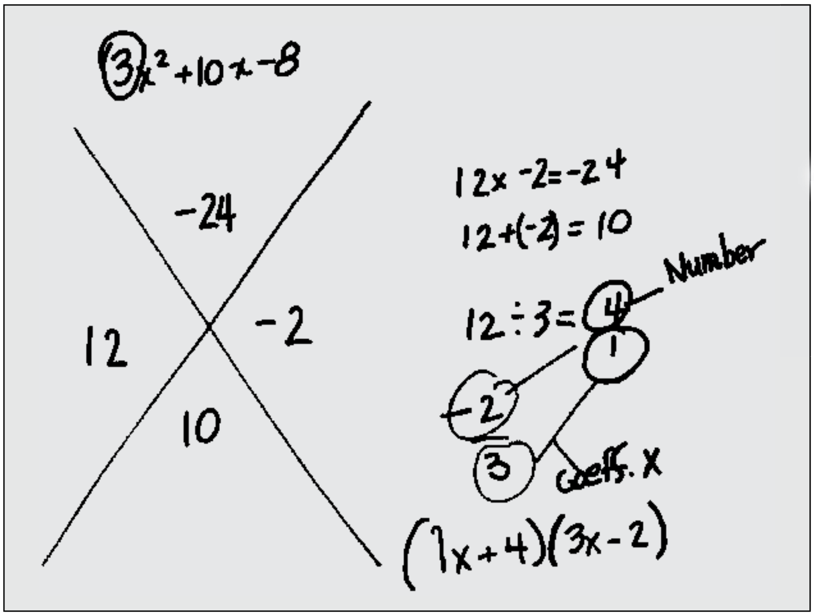


Fig. 2 Factoring using "the magic X" also detracts from students' understanding of what the underlying mathematics really is.

WE ADVOCATE FOR CONSISTENT USE OF APPROPRIATE RULES AND MATHEMATICAL VOCABULARY.

13. The Magic X

In some cases, students have been taught that to factor $3x^2 + 10\,x - 8$, they start with the symbol X on the paper. Then put the product of –3 and 8 in the top part of the X and the coefficient of the linear term in the bottom part of the X. Find two numbers that have a product of –24 and also sum to 10. In this case, it would be 12 and –2, which are both written in the left and right portions of the X (see **fig. 2**). Divide each of these two numbers by the coefficient of x^2. The answers would be

$$\frac{12}{3} \text{ and } -\frac{2}{3},$$

which should be simplified if possible. When simplified, the results are

$$\frac{4}{1} \text{ and } -\frac{2}{3}.$$

The denominators tell you the coefficients of x; the numerators tell you the constants $(1x + 4)(3x - 2)$. This rule may lead students to think that any expression can be factored. Additionally, the rule (or process) falls apart for factorable expressions when the degree is not 2.
Expiration date: High School (A.REI.4B)

14. You Cannot Factor a Polynomial, such as $x^2 - 5$.

Students are often told to look at the constant in a polynomial expression. If that constant is a prime number, the polynomial cannot be factored. The misconception that arises from this rule is that students then assume that such an equation as $x^2 - 5 = 0$ cannot be solved.
Expiration date: High School (A.REI.4B, N.CN.4)

15. Quadratic Equations Always Have Two Solutions, and Linear Equations Always Have One Solution.

The rule that students often learn is that if you are solving a quadratic equation, you will have two solutions because the exponent of the variable is 2. If solving a linear equation, you will only have one solution. Quadratics could have two solutions, or one solution if its vertex is on the x–axis, or zero solutions if it never intersects the x–axis. Additionally, linear equations could have one solution, or no solutions if, for example, the variable terms simplify to zero, such as $3x + 5 = 3x + 8$, or an infinite number of solutions when the variable can work

Table 1 Expired Vocabulary or Notation

What Is Stated	Appropropriate Alternatives
Solve an expression.	Evaluate an expression.
Show all your steps when solving an equation.	Show your thinking (use words, diagrams, graphs and so on).
Get rid of the . . . (fraction, decimal, coefficient, like term, and so on).	Using the terminology *get rid of* assumes that the indicated quantity disappears. This action, however, should instead be linked to a property of an operation or equality.
Plug in a value for a variable.	The word *plug* is not a mathematical term. The use of the word *substitute* more accurately depicts the action.
Reduce a fraction.	Use of the word *reduce* often indicates something is made smaller. That is not the case with equivalent fractions. Use the word *simplify* rather than *reduce.*

for any value, such as $2x + 4 = 2(x + 2)$. Expiration date: High School (F.IF.7)

MATHEMATICAL TERMINOLOGY

The mathematical representations we encourage students to use, including vocabulary, should be carefully considered. Using accurate and precise terminology (Common Core Standards for Mathematical Practice [SMP] 6) provides consistent development of the underlying concepts so students can focus on the new ideas rather than the changing landscape of vocabulary and symbolic representations. **Table 1** includes expired language that is heard or used in many secondary classrooms. In each case, we provide alternatives that will align better with the conceptual underpinnings students should develop.

SEAMLESS PROCEDURES AND CONCEPTUAL UNDERSTANDING

Teaching students in ways that promote conceptual understanding and procedural fluency so that students circumvent using rules that expire or vocabulary that is not mathematically correct is the job of mathematics teachers at all grade levels—we are in this together. In this article, we provide a nonexahustive list of 15 problematic mathematics rules that cause even high school students to stumble. We advocate for building a schoolwide plan or whole-school agreement (Karp, Bush, and Dougherty 2016) for the consistent use of appropriate rules and mathematical vocabulary. When teachers are intentionally consistent with vocabulary, diagrams and processes, students will be better able to build upon their foundational understandings as they progress throughout the grades. High school mathematics is complex and challenging enough as it is; we work to avoid rules that expire and instead aim to present mathematics in a seamless way that balances procedural fluency with conceptual understanding.

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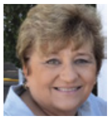

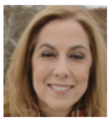

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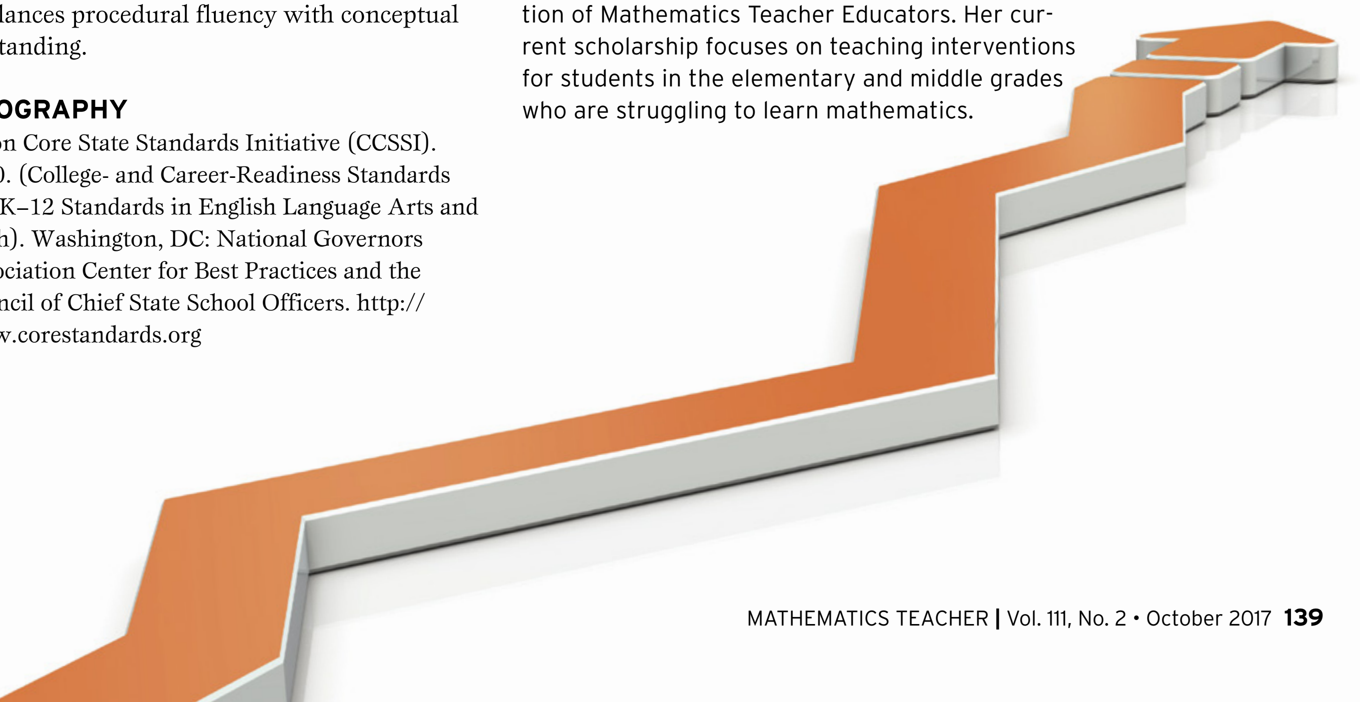

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